THE

MATHEMATICAL GAZETTE

EDITED BY

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GEOMETRY IN SECONDARY SCHOOLS.

By R. M. Carey.

After the recent papers on the teaching of Geometry by Professor Neville, I feel that it is almost impertinent, or worse, dull, to reopen the subject again so soon; and if this paper were concerned mainly with general principles, it would have little chance of adding much that is interesting and relevant. My concern, however, will be more with class-room work, and with advocating in detail a reform of the present syllabus; and when I have to talk of principles, by attacking the matter from a different angle, I hope to add a little that is new to the discussion that numbers Pythagoras and Prof. Neville amongst its distinguished protagonists.

In the following pages I shall try to carry some of the conclusions in "The Teaching of Geometry" one stage further, and suggest exactly what parts of the present syllabus of Secondary Schools should be omitted, and what retained. I feel a little ungracious in thus using the aid of Prof. Neville, since in certain important respects I differ considerably from him (and incidentally from Pythagoras as well!)

Perhaps I should start by describing my ideal of the geometry that can interest the average boy. Pascal's theorem is an admirable example; and the features to which I attach especial importance are first the simplicity and manifest truth of the result; it can be understood and verified by boys of 12: and next, the difficulty of the proof; showing that the bridge between straightforward matters such as the axioms and three points lying in a straight line must be built of high abstraction. In addition, quite a number of boys can appreciate the form of the proof; its unembellished strength and alternative methods showing the advantages of elegant work, and the possibility of arriving at the result by apparently unconnected methods. Indeed, mathematicians are fortunate

in possessing the only subject that seems suitable for showing the culture of Ancient Greece to the average boy ; Geometry, the most direct of the Sciences, gives the simplest example of the search for truth ; and as always Truth in its simple forms brings with it Beauty.

I think that most of my readers will agree that the present syllabus of Geometry in Schools has fallen sadly from the Greek ideal. It is seldom better than a welter of rather unrelated theorems, whose only apparent object is to cover the syllabus of some examination ; together with a collection of riders, at once shapeless and formidable. All too often it is little more than a cram in theorems and certain constructions that are likely to be set.

In order to obtain some form out of chaos, I should like to see something like the following : at the age of 12, or so, a class might discuss, and then verify by drawing and measurement, the spectacular theorems about the nine-point circle and Brianchon's theorem : they would then be told that their work during the next few terms would be directed towards the investigation of these theorems, and others like them. As a guess, 10 per cent. of the boys would not be interested, and they should be allowed to drop geometry in favour of gardening or fretwork. But most boys would be interested, and for them the syllabus should be chosen so as to lead to these results in the most simple and easy manner. If this is accepted—and it seems to me to replace an ill-assorted mass of theorems by a subject with a definite and noble aim—some drastic consequences follow : the chief is that the whole theory of congruent and equivalent triangles must be omitted, for none of the interesting results needs them.

This conclusion is unorthodox, and it is perhaps worth stating that I reached it by a different path ; I noticed that congruent triangles needed more drill than any other part of geometry, and that it was more easily forgotten by boys : the natural conclusion was that there was something very wrong either with my teaching or else with the subject. The next stage was the memory of a phrase about the proof of the congruent triangle theorems " a tissue of nonsense "; and finally came a search for the results obtained by the use of congruent triangles. This last revealed that all profitable results can be established by symmetry, a much shorter and more convincing method, to men and to boys.

The study of equivalent triangles, too, fulfils no useful purpose; Pythagoras' theorem can be established by other means ; the Euclidean proof is magnificent, and should be shown to all clever boys, say two hours devoted to it at about the age of 16, in the same spirit that one visits a museum of Assyrian treasures—impressive creations, but dead and with no direct posterity. Equivalent triangles provoke many remarks from boys, " I don't see the point of it " ; and neither do I ! I believe that my readers will agree that boys normally revel in the more objective theorems, angle properties of circles, Pythagoras, etc., in marked contrast to their puzzled reluctance in dealing with equivalence and congruence.

These important omissions have justified a full enquiry; but there are others with which I shall deal more briefly.

In the first place some further barren work:

Only one circle can be drawn through three points.
Tangent and radius are perpendicular.
Inequalities.
If three sides are proportional, the triangles are similar (the same with two sides and the included angle).
Ratio of areas of similar triangles.
The angle in the major segment is obtuse.

Some real nonsense:

Geometrical identities.
Proofs of congruence.

Proofs that are obvious by symmetry or common sense:

Perpendicular on base of isosceles triangle bisects it.
Equal arcs subtend equal angles at the centre.
Perpendicular from centre bisects chord.
If two triangles are equiangular, their corresponding sides are proportional.
If a line is parallel to the base of a triangle, it cuts the sides proportionally.
The intercept theorem.

The following should be reserved for able boys only:

Extensions of Pythagoras, and Apollonius' theorem.
Equivalent figures.

Were these alterations made in the syllabus, all boys would get further than they now do; not only is a great deal cut out, riders as well as theorems, but much of it is calculated to blunt the geometrical intelligence. At present the average boy gets a working knowledge of the circle and similar triangles, but has little time to work with the combination; but the fusion of two very different ideas is almost invariably most profitable; and here, just beyond the present leaving standard, lie a great number of marvellous theorems which are within the reach of the ordinary boy provided that his time and geometrical desire are not wasted as they now are. Looked at from another angle, congruence, equivalence and other unsuitable matters account for the mixture of incapacity and dislike for geometry shown by many boys of proved ability and keenness in Classics, English, and even Algebra.

In a most stimulating paper, published in the *Gazette*, October 1932, Mr. Andersen pointed out the danger of setting boys to work on an uncharted sea of riders. He showed the advantages of allowing boys to spend some of their time in finding out geometrical truths for themselves, and gave most valuable constructive suggestions as to how to set about this. I should like to add one or two remarks, for I agree wholeheartedly that there is serious danger of boys finding mere rider solving monotonous and pointless. The first is that, by including theorems such as Pascal's, geometry would fall into groups, each group leading to a notable result; *e.g.* Menelaus and Ceva's theorems, leading on to Pascal, including the necessary

riders, would take a week or ten days; inversion, another ten days. Thus practice and skill in rider solving would fall into its proper place as the means towards an end; the present tendency is all means, no end.

In the next place, proper choice of riders can, I think, do much to make the proof worth doing. Consider for example the following results of riders:

(i) AB is parallel to CD;
(ii) $\angle ABE = \angle X$;
(iii) $\angle ADK + \angle C = 180°$;
(iv) $AX^2 + AN \,.\, PQ = AX \,.\, CD$.

Riders could be constructed so that No. (i) was the hardest, or alternatively the easiest; but I am not concerned with the aspect of difficulty. They are arranged in order of the simplicity and desirability of the result.

No. (i) is the type of Brianchon's theorem; easily verifiable, possibly by a rough figure, and this brings out in sharp contrast the necessity for a difficult theoretical proof.

No. (ii) is simple and easily checked, but not so obviously worth proving; after all, in most figures one can pick out several pairs of equal angles.

No. (iii) is verifiable, much less simply; but as far as a boy is concerned hardly worth proving; after all, $\angle D + \angle C$ have to add up to something or other.

No. (iv) is complicated and meaningless; whilst verification needs the bludgeon-like method of long multiplication, where approximation will blunt the fine edge of truth and wreck its beauty.

The conclusion to be drawn is that the ideal collection of riders should have all their "To Proves" simple, objective results such as No. (i); but in practice, it is enough that there should be a sprinkling through the book, and especially at the end of each year's geometrical course. And whilst during the early part of the course riders should be given in the didactic form:

ABCD is a cyclic quadrilateral, in which . . . ; prove that X is the mid-point of CD.

In a later stage there is much to recommend the impersonal:

ABCD is a cyclic quadrilateral, . . . ; then X is the mid-point of CD.

In the clearer form given above the boy is obeying the command of a schoolmaster or a textbook writer; in the latter he is investigating abstract truth.

The course of elementary geometry that I should like to see, in Stage B, would be as follows:

Part I. The study of parallel lines, isosceles triangles, properties of parallelograms, Pythagoras, angle at centre of a circle, angles in the same segment in semicircle, and opposite angles of cyclic quadrilaterals, perpendicularity of tangent and radius, equality of tangents.

Part II. Numerical examples on equal areas, and on similar

triangles, loci, symmetry, the exterior angle of a cyclic quadrilateral, the alternate segment theorem, equal arcs, conditions for four concyclic points.

Part III. A grand attack on the four theorems concerned with similar triangles (similar triangles, proportionals, chords, bisector of angle of triangle).

Part IV. The nine-point circle, Euler's line, Apollonius' circle, inversion, Menelaus' and Ceva's theorems, collinearity and concurrency, properties of the triangle, orthogonal circles.

In each part, the result of the theorem should be learnt by drill examples, mainly numerical, before giving theoretical riders; and finally, in each part, a collection of riders whose solution depends on any one of all the theorems in the group should be given.

It is almost impossible to overstress the importance of careful grading in respect of difficulty for Parts I and II; at every new theorem quite a number of figures should be drawn, in the book or on the board, so that slow boys can manage three or more riders in a lesson instead of one and the figure of the second. Moreover, it is a good plan for the teacher to note in the margin of his textbook, against each rider *eee* (meaning 3 easy steps), *eh* (meaning 1 easy, 1 hard step); *e* would be used for angles in an isosceles triangle, angles in the same segment, alternate angles, etc.: *h* applies to steps such as allied angles, exterior angle of cyclic quadrilaterals, angles in the alternate segment, and other theorems that are harder to visualise.

Part II is similar to Part I; my own bias would lead me to give rather fewer numerical examples, and a number of revision examples on all back work. Symmetry is perhaps best illustrated by an example:

In the triangle ABC, $AB=AC$ and X is any point in AB; DXE drawn parallel to BC cuts the circle ABC at D, E; prove that $\angle BDE=\angle CAD$.

This rider would be classified as *ee*, provided sufficient drill examples on symmetry had been worked.

For many boys Parts I and II might be sufficient; but if Part III is attacked, it must involve very thorough training in work of a much higher degree of abstraction, namely the ratio $AB:XY$, $\left(\text{which the boy must invariably write in the form } \frac{AB}{XY}\right)$, and the product $AB\,.\,XY$. This necessitates many undesirable riders such as "Prove that $CD\,.\,HK=LM\,.\,EF$": but after even a fortnight, several riders proving AB is parallel to PQ can be given, or followed on the board. The greater abstraction of the method results in fewer concrete riders, but also gives greater satisfaction when a rider is solved. One further consequence is that it rules out many boys from taking the course with profit.

One final note; is it not time that all talk of proving the rectangle $CD\,.\,HK=$ rectangle $LM\,.\,EF$ should be dropped? The essential antithesis is between the product and the ratio; there is no advantage in trying to achieve a false practical basis, by clinging blindly

to tradition. It is brought out sufficiently by the one and only rider : " Draw a square equivalent to a given rectangle " : by overstressing, we create a complete muddle between the fundamentals, the ratio $AB:XY$ and the rectangle $AB.XY$. (Why rectangle, Sir ?)

As to Part IV, I have no intention that boys should do all of it, though it is within the ability of the average boy who passes the School Certificate. It is the reward for their labours, the consolidation of their knowledge and the proof that the geometrical course has not been in vain. Its essential lies in simple concrete results, as opposed to a conception such as the self-polar triangle, which is suited more to the specialist-to-be ; and for him this last is an excellent stimulus, leading towards higher abstraction.

This paper is already too long to allow me to make any definite suggestions about examinations ; in general, I should like to plead for fewer theorems, and more rider questions, not necessarily of greater difficulty, though often using circle and similar triangle theorems in one question. This seems to me to be the best way out of the present situation, which is far from satisfactory ; for it is now possible to achieve success by scamping the riders and cramming the wide syllabus of theorems. But the value of the subject lies in the opposite direction : the method of geometrical proof may be mastered in a very restricted field ; but in that field the solution of plenty of riders is essential. R. M. C.

GLEANINGS FAR AND NEAR.

982. He saw his life as a problem in higher mathematics, the working out of which had required intense application of all his powers, but of which the result had not the least practical consequence.—Somerset Maugham, " Back of Beyond ", *Ah King* (1933). [Per Miss A. E. Leake.]

983. They [the Whigs and the Tories] bear some analogy to the two forces which retain the planetary bodies in their orbits ; the annihilation of one would disperse them into chaos, that of the other would drag them to a centre. —H. Hallam, *The Constitutional History of England* (1855), iii, p. 201.

When I was about nine years old I was taken to hear a course of lectures, given by an itinerant lecturer in a country town . . . there was the excessive notion of creative power exhibited in the millions of miles of the solar system, of which power I wondered they did not give a still grander idea by expressing the distance in inches. But even this was nothing to the ingenious contrivance of the centrifugal force. " You have heard what I have said of the wonderful centripetal force, by which Divine Wisdom has retained the planets in their orbits round the sun. But, ladies and gentlemen, it must be clear to you that if there were no other force in action, this centripetal force would draw our earth and the other planets into the sun, and universal ruin would ensue. To prevent such a catastrophe, the same wisdom has implanted a centrifugal force of the same amount and directly opposite," etc. I had never heard of Alfonso X of Castile, but I ventured to think that if Divine Wisdom had just let the planets alone it would have come to the same thing, with equal and opposite troubles saved.—A. De. Morgan, *Budget of Paradoxes*, (1915), ii, p. 269.

INTRODUCTORY THEOREMS IN GEOMETRICAL CONICS.

By B. E. Lawrence.

1. Nowadays the beginning of conics is a blend of pure and coordinate geometry, but the starting point is generally the focus-directrix definition. From this the standard equations and properties are deduced, but a good deal of time is taken before some of the more powerful methods can be applied.

The object here is to indicate compact proofs of some well-known results, and to obtain, at an early stage, such results as Newton's Theorem and the cross-ratio property from the focus-directrix definition.

The treatment is by elementary geometry, and makes use of a circle with its centre at the focus.

We begin with a result of which Adams' property is the limiting case; at the end, a simple analytical application of this result is given.

1·1. *If T is any point on a chord PQ of a conic which meets the directrix in Z, and if parallels are drawn through T to ZS and the axis to meet SP and the directrix in P_1 and N respectively, then*

$$SP_1 = eTN.$$

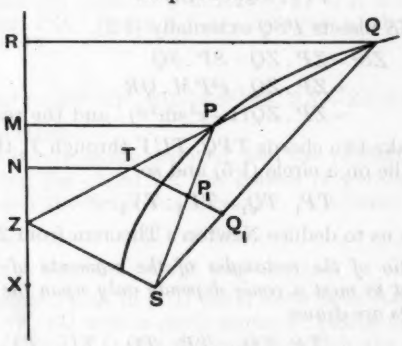

Fig. 1.

For
$$\frac{SP_1}{SP} = \frac{ZT}{ZP} = \frac{TN}{PM},$$
and so
$$\frac{SP_1}{TN} = \frac{SP}{PM} = e.$$

We deduce at once

1·2. *SZ bisects PSQ externally.*

For $SQ_1 = eTN = SP_1$, and so P_1Q_1 is equally inclined to SP, SQ.

1·3. *If TP is a tangent meeting the directrix in Z, SZ is perpendicular to SP.*

1·4. *If TP is a tangent and L is the projection of T on SP, $SL = eTN$.* [Adams' property: hence follow the tangent properties and the construction for tangents from a point.]

The above are well known: the next is trivial but is applied in what follows.

1·5. *The locus of P_1, Q_1, when T is fixed, or moves parallel to the directrix, is a circle* (centre S, radius eTN).

Note in particular that, if T is on the latus-rectum, the circle has the latus-rectum as diameter.

The circle will be referred to as the tangent circle.

2. In proving our next result we use the theorem that if the external bisector of the angle A of a triangle meets BC in Z, then

$$ZA^2 = ZB \,.\, ZC - AB \,.\, AC.$$

2·1. *If PQ makes an angle θ with the directrix,*

$$\frac{TP \,.\, TQ}{TP_1 \,.\, TQ_1} = \frac{1}{1 - e^2 \sin^2 \theta}.$$

For $$\frac{TP}{TP_1} = \frac{ZP}{ZS}, \quad \text{and} \quad \frac{TQ}{TQ_1} = \frac{ZQ}{ZS}.$$

Hence $$\frac{TP \,.\, TQ}{TP_1 \,.\, TQ_1} = \frac{ZP \,.\, ZQ}{ZS^2}.$$

But, since ZS bisects PSQ externally (1·2),

$$\begin{aligned} ZS^2 &= ZP \,.\, ZQ - SP \,.\, SQ \\ &= ZP \,.\, ZQ - e^2 PM \,.\, QR \\ &= ZP \,.\, ZQ (1 - e^2 \sin^2 \theta) \quad \text{and the result follows.} \end{aligned}$$

If we now take two chords TPQ, TUV through T, the four points P_1, Q_1, U_1, V_1 lie on a circle (1·5) and so

$$TP_1 \,.\, TQ_1 = TU_1 \,.\, TV_1.$$

This enables us to deduce Newton's Theorem from 2·1.

2·2. *The ratio of the rectangles of the segments of chords drawn through a point to meet a conic depends only upon the directions in which the chords are drawn.*

For $$\begin{aligned} \frac{TP \,.\, TQ}{TU \,.\, TV} &= \frac{TP_1 \,.\, TQ_1}{1 - e^2 \sin^2 \theta} \Big/ \frac{TU_1 \,.\, TV_1}{1 - e^2 \sin^2 \phi} \\ &= \frac{1 - e^2 \sin^2 \phi}{1 - e^2 \sin^2 \theta}. \end{aligned}$$

From this result we may, reversing the usual order in pure geometry, deduce Carnot's Theorem.

2·3. *If the sides BC, CA, AB of a triangle cut a conic in A_1, A_2; B_1, B_2; C_1, C_2, respectively, then*

$$AB_1 \,.\, AB_2 \,.\, CA_1 \,.\, CA_2 \,.\, BC_1 \,.\, BC_2 = AC_1 \,.\, AC_2 \,.\, BA_1 \,.\, BA_2 \,.\, CB_1 \,.\, CB_2.$$

Let the sides make angles θ, ϕ, ψ with the directrix.

Then $$\frac{AB_1 \,.\, AB_2}{AC_1 \,.\, AC_2} = \frac{1 - e^2 \sin^2 \phi}{1 - e^2 \sin^2 \psi}.$$

Multiplying together three results of this kind, the product on the right-hand side is unity.

From these two theorems a great many other properties follow, but these will not be considered. Two other immediate consequences of 2·1 may be mentioned.

2·3. *If TPQ, TUV are perpendicular chords* $\dfrac{1}{TP\,.\,TQ} + \dfrac{1}{TU\,.\,TV}$ *is constant for all pairs of such chords through T.*

For $$\frac{TP_1 \,.\, TQ_1}{TP \,.\, TQ} + \frac{TU_1 \,.\, TV_1}{TU \,.\, TV} = 1 - e^2 \sin^2 \theta + 1 - e^2 \sin^2 (\theta + \tfrac{1}{2}\pi) = 2 - e^2,$$

and $$TP_1 \,.\, TQ_1 = TU_1 \,.\, TV_1 = \text{const.}$$

We can also deduce quickly that the curve has a "centre" by finding the locus of the middle points of a system of parallel chords. Only the result leading to this will be proved, because there are better methods of dealing with the centre.

2·4. *If SY is drawn perpendicular to the chord PQ whose middle point is V,*
$$YV/ZV = e^2 \sin^2 \theta.$$

Let ZS meet the circle SPQ in K. By (1·1) K is the mid-point of one of the arcs PQ. Hence KV is a diameter of the circle SPQ and is parallel to SY.

By (2·1),
$$1 - e^2 \sin^2 \theta = ZS^2/ZP \,.\, ZQ = ZS^2/ZS \,.\, ZK = ZS/ZK = ZY/ZV.$$

Hence the result.

The locus is now easily deduced, and the existence of the centre by considering the chord (of the system of parallels) through the point in which the locus cuts the axis. It is found that
$$XC = SX/(1 - e^2),$$
and so there is a centre, except when $e = 1$.

3. The points P_1, Q_1 in (1·1) could have been found as the intersections of SP, SQ with a circle centre S, radius eTN (incidentally this method gives P_1Q_1 for the case in which the original method fails, viz. when PQ is a focal chord).

If a circle of radius l is drawn round the focus, it will cut SP, SQ in p, q, say, pq will cut ST in t, and we shall have
$$\frac{ST}{St} = \frac{SP_1}{Sp} = \frac{P_1Q_1}{pq} = \frac{e \,.\, TN}{l},$$
i.e. we have drawn the same figure on a different scale. Choose l to be the semi-latus-rectum and take points A, B, C, ... on the conic and call the corresponding points on the circle a, b, c, ... T may be regarded as the point in which any one of the chords AB, BC, ... meets the latus-rectum and so, by 1·5, all the pairs AB, ab, BC, bc, ...

meet on the latus-rectum. We obtain in fact two figures with S as centre and the latus-rectum as axis of perspective.

Using this fact we might deduce theorems about conics (*e.g.* Pascal's Theorem) from theorems about circles. The cross-ratio property of the conic will be sufficient illustration of the method.

3·1. *If A, B, C, D are fixed points on a conic and P a variable point, then $P(ABCD)$ is constant.*

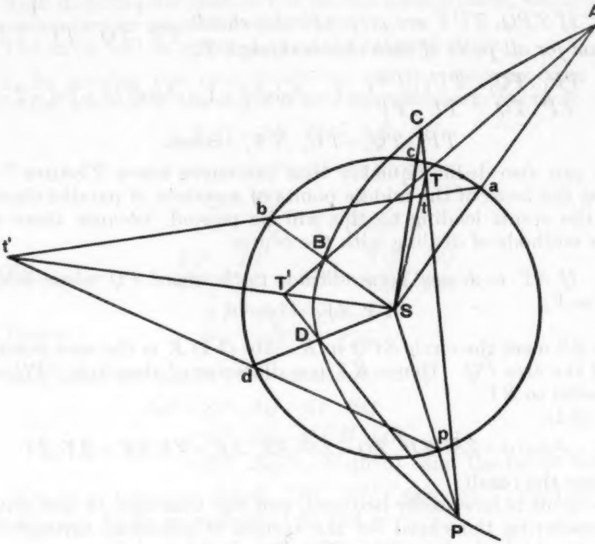

FIG. 2.

First notice that from the first line of the proof of 2·1,

$$\frac{TP}{TQ}=\frac{TP_1}{TQ_1}\cdot\frac{ZP}{ZQ}.$$

Now if we take pqt instead of P_1Q_1T,

$$TP_1/TQ_1=tp/tq,$$

and so

$$\frac{TP}{TQ}=\frac{tp}{tq}\cdot\frac{ZP}{ZQ}.$$

Now let AB cut PC, PD in T, T' and let the circle on the latus-rectum as diameter cut SA, SB, SC, SD, SP in a, b, c, d, p. Finally, let ab cut ST, ST' in t, t'.

We notice that pdt', pct are straight lines. If this is not obvious from the construction of the figures, draw $T'B'A'$ parallel to $t'ba$ to cut Sb, Sa in $B'A'$. Then

$$St'/ST'=Sb/SB'=l/eT'N' \text{ by } 1{\cdot}1,$$

i.e. St'/ST' is independent of the direction $t'ba$. Hence pd cuts ST' at t'.

Now $\dfrac{TA}{TB}=\dfrac{ta}{tb}\cdot\dfrac{ZA}{ZB}$ and $\dfrac{T'A}{T'B}=\dfrac{t'a}{t'b}\cdot\dfrac{ZA}{ZB}$, by the remark above.

Hence $(AB, TT')=(ab, tt')$ by division.

Therefore $P(AB, CD)=p(ab, cd)$.

But $p(ab, cd)$ is constant by the angle properties of a circle.

Hence $P(AB, CD)$ is also constant.

4. Finally, we may note that (1·1) and (1·4), Adams' property, lead immediately to two well-known equations in polar coordinates.

Suppose the vectorial angles of P, Q are as usual $\alpha\pm\beta$ and T is the point (r, θ).

Then in (1·4), put $SL=r\cos(\alpha-\theta)$,

and $TN=SX-r\cos\theta=l/e-r\cos\theta$,

and we get at once the equation of the tangent.

4·1. $$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha).$$

In 1·1 projecting at right angles to P_1Q_1, we have

$$SP_1\cos\beta=r\cos(\alpha-\theta).$$

Also $SP_1=eTN=l-er\cos\theta$, as before.

Combining these, we get the equation of the chord

4·2. $$\frac{l}{r}=e\cos\theta+\sec\beta\cos(\theta-\alpha).$$

The equation of the polar of a point may also be found by a similar geometrical method.

B. E. L.

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984. His mind began challenging the enigma of that other scholar, that Gebert who also became Pope, though for three years only, till he died. Necromancer, devil-aided, devil-destroyed, said the legend, and all, Thierry of Chartres used to say, because he had a head for mathematics, and had studied Arabic and geometry at the schools of the Saracens. It was hard to come at any truth about him: but there, Abelard had always felt, was a man with whom he would have been on terms.—Helen Waddell, *Peter Abelard*, p. 8. [Per Mr. E. H. Lockwood.]

985. I was once acquainted with a gentleman who was a great mathematician. Whenever I was in company with him he always used the same expressions, which differed very little from geometrical problems. When he was asked if he chose cream in his tea, this was his constant answer: "Yes, Ma'am; because the globular particles of the cream render the acute angles of the tea more obtuse". This reply might be tolerably well received for the first time; but from the repetition, and being often ill-timed, disgusted.—Ann Murry, *Mentoria; or the Young Ladies Instructor* (1778). [Per Mr. F. Beames.]

SOME IDEAS ON ENERGY AND MOMENTUM.*

BY H. E. PIGGOTT.

THIS paper is divided into three parts. The first is historical—a brief survey of the growth of Kinetics, with special reference to ideas on momentum and energy; the second part is mainly pedagogic—some rather disjointed remarks on points that arise in teaching, together with a few worked examples; in the third part I take the special problem of direct impact and subject it to a closer mathematical analysis than that found in text-books.

I.

I hope we all cherish the ideal, even though all of us do not put it into practice, that in teaching any branch of mathematics, we should have a historical background to our knowledge. Not that it is always advisable or indeed possible to follow the historical order in presenting a subject, nor are there many opportunities of making historical references in the course of our teaching, though often such are very effective in reawakening the interest of a flagging class. The individual learns much as the race has learnt. Generally speaking, the places in the development of a mathematical theory at which the boy or girl of to-day sticks, are just the places at which the race has stuck. An analysis of these difficult fences and of the manner in which they were surmounted helps us, I think, to greater understanding and greater patience. To put it on more philosophical grounds, here is a quotation:

"The historical investigation of the development of a Science is most needful lest the principles treasured up in it become a system of half-understood prescripts, or, worse, a system of prejudices. Historical investigation not only promotes the understanding of that which now is, but also brings new possibilities before us, by showing that which exists to be in great measure conventional and accidental. From the higher point of view at which the different paths converge, we may look about us with freer powers of vision, and discover routes before unknown."

That was a quotation from Mach's *The Science of Mechanics*, and this book has been the main source from which I have drawn the material for the remarks which follow. I make no apology for serving up a *réchauffé* of Mach, because I do not think that this great work is much read nowadays. It is rather difficult reading because of its German thoroughness and its nineteenth-century leisureliness. It needs quarrying, and modern conditions do not leave many of us time to quarry.

The history of the development of elementary dynamical theory centres mainly round the names of Galileo, Huyghens, and Newton. It is a commonplace that Galileo distrusted the Aristotelian ideas relating to falling bodies, from which it resulted that a heavy body

* This paper was read to the London Branch of the M.A. on February 24th, 1934.

would fall a given distance in a shorter time than a lighter one, and that he had the hardihood to test and disprove the theory by the experiment of dropping different weights simultaneously from the leaning tower of Pisa. But his subsequent procedure may not be so clearly understood. He asked himself the question, with what velocity will a body falling from a given height strike the earth? Here he was up against the difficulty which still confronts those who study elementary dynamics. How can we, in practice, measure the velocity at an instant when a body is accelerating? Galileo recognised that it could not be done directly. All he could do was to assume that the velocity followed some simple rational law, to make a shot at such a law, to find out some logical consequences capable of experimental treatment and then to test these. His first shot was that the velocity varies as the distance fallen. He disproved this by a logical argument which is fallacious. It runs as follows: a body falls h feet and acquires a velocity of v; a second body falls $2h$ feet and acquires a velocity of $2v$. Therefore the times of falling are the same; therefore the second body describes the second half of its fall in no time at all, which is absurd. The fallacy is, of course, due to a loose conception of average velocity. It is an example of a little knowledge not being a dangerous thing. If Galileo had known that his assumption led to the law that s varies as e^{kt} he might have been held up for quite a long time. But, satisfied with his argument, he made a second shot, this time the lucky one that v varies as t. From this he deduced logically that s varies as t^2. The argument is familiar to us. He drew the straight line we now call the velocity-time graph, inserted the ordinate at the middle of the interval and pairs of ordinates equidistant from that on either side of it. He then argued that the distance fallen would be the mean velocity (given by the middle ordinate) multiplied by the time. This was a great achievement for that date; it is really an example of simple graphical integration and we now serve it up in exactly Galileo's form to young students. There was now some hope that he could put the law to the test. But there were obvious difficulties in the way of testing it for bodies falling vertically. But he argued again, quite correctly, that the same type of law must hold for bodies sliding or rolling down an inclined plane. (The argument may be found in Mach or in Cox's *Dynamics*.) Here was something he could test. He made a groove down an inclined plane and marked off distances from the top end proportional to 1, 4, 9, 16, 25. A marble was rolled down the groove and the time taken as it passed each mark. These times were found to be proportional to 1, 2, 3, 4, 5. So his law was established. Galileo had, of course, no stop-watch, so he got over the difficulty by the use of a water-clock. A vessel of large cross-section, filled with water, had a small orifice in its base. This was kept closed by the finger. When the finger was removed for a short time, a few drops of water flowed out on to the scale of a balance. Galileo later substituted a pendulum for a water-clock. He thus established the laws $v=gt$ and $s=\frac{1}{2}gt^2$ for falling bodies together with the concept of uniform acceleration. He does not

seem to have deduced the third law $gs = \frac{1}{2}v^2$; at any rate he did not interpret it.

"Note", says Mach, "that it is an anachronism to attempt to derive the uniformly accelerated motion of falling bodies from the constant action of the force of gravity. Such an exposition is unhistorical and places Galileo's discovery in the wrong light. The whole notion of force as we know it to-day harks back to Galileo's work. No-one can know, who has not learnt it from experience, in what manner Pressure (as Galileo called Statical force) passes into motion, that not position nor velocity but acceleration is determined by it".

Next comes Huyghens, less brilliant as a philosopher than Galileo, but a more competent mathematician. He followed up Galileo's work on the simple pendulum, and extended it to the compound pendulum and to systems of masses. Using the equation $gs = \frac{1}{2}v^2$, ignored by Galileo, he finally arrived at

$$\Sigma mh = \frac{1}{g}\Sigma \tfrac{1}{2}mv^2,$$

that is to say, the concept of work as determinative of velocity. Huyghens' achievements were largely ignored or distrusted by his contemporaries. He was a pioneer in other respects; he used the theory of the pendulum as the basis of experiments to determine the value of the acceleration due to gravity, he worked out the theory of circular motion, he invented and constructed the pendulum clock and the escapement, and he published the theory of impact eighteen years before the *Principia*. Most of the problems he dealt with were complex, and his manner of exposition often obscure. Hence the contemporary neglect of his work on dynamical theory.

Next we come to the great Newton, a philosopher as well as a mathematician of the front rank. His object was to "know Nature", to investigate and transform the expression of facts, not to frame hypotheses. In addition to the discovery of the law of Universal Gravitation he made the following chief advances in Mechanics: (i) the generalisation of the idea of force, (ii) the introduction of the concept of mass, (iii) the statement of the law of action and reaction. He cleared up the idea of Inertia, latent but unexpressed in the work of Galileo.

It was only slowly and with difficulty that the concept of work attained its present position of importance. Whenever it appeared, it was sought to replace it by the earlier ideas of force and momentum. To put it into modern terminology, the classical method—that of the line Galileo-Newton—is to think of work as the space-integral of a force. The method of Huyghens, if logically pursued, leads to the concept of force as the space-differential of a work-function. Mechanics could have been built up on either line. What prevented Huyghens from seeing the full implications of his researches was that he had no clear concept of mass. Newton either distrusted the work of Huyghens or was ignorant of it.

It was noticed shortly after the time of Galileo that there is, inherent in the velocity of a body, something which corresponds to

force, some "efficacy" by means of which a force can be overcome. "What", they asked, "is the nature of this efficacy; how is it measured?" It must be remembered that the early terminology of Mechanics was very loose. Galileo used the terms "impulse", "momentum" and "force" almost indiscriminately. There was, it seems, a tendency when in doubt to take shelter behind the word *vis*, a term which lacked precision. Thus Newton used *vis insita* for inertia, and, in the statement of the Second Law, *vis motrica impressa* for impressed force. *Vis mortua* was used for statical force, or "pressure" as it was generally called. But what was *vis viva*, the "efficacy", akin to force, but due to motion? Descartes, in 1644, said that it must be the same as momentum, measured by mv. His argument depended on what we now call the velocity-ratio of machines. He went on to say that the sum-total of the momenta in the universe remains constant. For this statement he rested his arguments not on experiments but on his theological prepossessions. He talks about the perfection of God, and continues, "Therefore, it is wholly rational to assume that God, since in the creation of matter He imparted different motion to its parts, and preserves all matter in the same way and conditions in which He created it, so He similarly preserves in it the same quantity of motion". Fortunately, nowadays, we keep our mathematics and theology in different compartments, to the clarification, at any rate, of mathematics.

Leibnitz in 1686 attacked the views of Descartes. He showed the fallacy in Descartes' velocity-ratio argument and, following Huyghens, gave mv^2 as the true value of *vis viva*. He went on to say that Σmv^2 is constant throughout the universe.

Descartes and Leibnitz mixed up two things in their famous controversy, the measure of the efficacy of motion and the constancy of the sums Σmv and Σmv^2. The controversy between the adherents of the two points of view lasted for 57 years; it was finally cleared up by d'Alembert in 1743. He pointed out that in one sense both were right. If the velocity of a body is doubled, it will be brought to rest by a given force in double the time but in four times the distance. With regard to the constancy of the sums, for a given material system not acted on by external forces, the Cartesian sum Σmv is constant; whereas, as Huyghens showed, Σmv^2 is constant, provided that work done by forces does not change it. When, therefore, our young pupils find it hard to understand "when to use momentum and when to use energy", it may give us added patience to remember that the race stuck at this point for 57 years.

Leibnitz, then, first fixed *vis viva* as the term for mv^2. Coriolis later used it to mean $\frac{1}{2}mv^2$, and the confusion has persisted. We have got over the difficulty by dropping the term and using kinetic energy for $\frac{1}{2}mv^2$. The word *energy* is due to Young in 1807. John Bernoulli first used the term the "conservation" of *vis viva*. He meant that when the *vis viva* disappears, *e.g.* in a pendulum at the end of its swing, the "facultas agendi" is not annihilated, but changed into what we call potential energy. The law was extended by Lagrange.

The dynamical principle of energy was extended to light by Fresnel in 1823. He assumed that the amount of *vis viva* brought per second to the refracting surface was equal to the sum of the *vis vivas* carried away by the reflected and refracted rays together. Joule in the 1840's extended the idea to heat, and the Conservation of Energy as a universal principle was first stated by Helmholz in 1847. The word *impulse* for the product $P \times t$, was proposed by Beranger in 1847.

I do not propose to follow up the changes in the idea of energy since the Theory of Relativity. These can be found in a paper by Professor Whittaker in the *Mathematical Gazette* for April, 1929. The final conclusion is that energy is no longer a scalar quantity, but one vector-component of an energy-tensor of which the other three components are momenta. The theorem of the Conservation of Energy comes down to the brief statement that the divergence of the energy-tensor is null.

II.

In the December number of the *Mathematical Gazette*, there appears a stimulating paper by Mr. G. H. Grattan-Guinness called "The Question of the Momentum" which is relevant to my subject. He starts with a sentence in inverted commas: "And so the impulse of the blow is equal to the change in momentum of the body", and continues as follows: "This phrase, coming at the conclusion of a more or less standard lesson in Elementary Mechanics, must surely have struck many teachers with a feeling of shame for having participated in a piece of not entirely straightforward work, and their pupils with an idea that they have witnessed some downright wangling. The usual impulse-momentum treatment cannot but be unsatisfactory to all concerned in the lesson, for it follows on the introduction of a new quantity of measurement of which, as so many of the text-books plaintively remark, 'no special name has been given to the unit'. If it is of so little importance that we do not even trouble to name its units, why bother with it? And why, in heaven's name, measure 'the effect of a blow' by the product of the force of the blow and some measurement of time?—the last thing that a beginner would expect, after some preliminary treatment of static and other forces measured in the ordinary way. Why bring in *time*? The general method seems to be to make a bald statement that Momentum is a quantity which is measured by the product of the mass and the velocity, and later to define Impulse as the product of the resisting force and the time of duration of the impact. It is easy, of course, to demonstrate the equivalence of these two quantities, but it seems without reason to have introduced either of them. One school text-book says: 'The dynamical relations of a moving body are usually expressed conveniently in terms of momentum'".

It is, of course, most regrettable that school text-books are still in use which employ such a phrase as "the dynamical relations of a moving body" being "usually expressed conveniently in terms of

momentum "—language which harks back to the fuzziness of thought of the followers of Descartes. With regard to the "plaintive note" that "no special name has been given to the unit of momentum", that is untrue. When the International Committee sat (in 1873) to settle the question of physical units, they invented a curious Esperanto for the absolute units in the C.G.S. scale. Of the terms so invented, *dyne* and *erg* are still in common use. But they also invented *bole* for the unit of momentum and *spoud* for that of acceleration. One of the difficulties in teaching Mechanics is, I think, just this, that we are dealing with terms like force, energy, impulse, momentum, which are used in ordinary non-technical speech, and by the popular journalist, with hazy meanings. It is most important to clarify and fix these meanings. No course on elementary dynamics which goes beyond mere kinematics can possibly keep out the idea of momentum. I do not share Mr. Grattan-Guinness' objection to making a bald statement as to the definition of the term at quite an early stage. I think there is a danger of trying to be over-logical in the presentation of the subject. But I would carry the idea a stage further. Mr. Grattan-Guinness uses *impulse* throughout only with reference to blows or impacts. This, as we have seen, is unhistorical, and I think it adds to the difficulties. Why not define Impulse as the product of a uniform force and the time during which it acts, both being measurable in terms of ordinary units?

The impulse-equation $P.t=m(v-u)$ $\left[\text{or } \frac{W}{g}(v-u)\right]$ follows as a symbolic form of the Second Law of Motion. Any experimental illustration of the Second Law will also verify this. To revert to the "plaintive remark" that no name has been given to the unit of momentum, it is now clear that, on the Hospitalier system, the unit is sec.-lb., just as the unit for kinetic energy is ft.-lb. (This unit is recommended in the M. A. Report on the Teaching of Mechanics.)

If the impulse equation is introduced in connection with the second law and examples worked on it, we can proceed, as Sir George Greenhill remarked, "from a push to a shove; from a shove to a blow". "We must take care", he adds, "to distinguish between them as in an action for assault and battery". The crux is, what happens in a blow? There are text-books which say that "an impulsive blow is caused by an infinite force acting through an infinitesimal time". We are careful, nowadays, in teaching Pure Mathematics, not to misuse terms like "infinite" and "infinitesimal". We no longer speak of the sum to an infinite number of terms of a geometrical progression, nor do we say that 1 divided by zero equals infinity, or that zero multiplied by infinity has any meaning or any value. Should we not be just as careful of expressing ourselves in Mechanics? In an impulsive blow, the force becomes very considerable, but not infinite, the time becomes very small—chronoscopically small, if I may coin the word—but not infinitesimal.

To revert to the paper I have been discussing, the author gives an

order of treatment which leads up naturally to the conservation of momentum. Finally he writes: "Now, we may say, in so short an interval of time, it is very difficult to measure the sudden resistance, R units, and it is equally difficult to measure the interval t units itself; but R and t, though themselves more or less immeasurable, can be measured together in product . . . This quantity is equivalent to $M(v-u)$ . . . etc.". This is all admirable, but I have one comment to make. The elementary mechanics we teach is an ideal science, in that we tend to assume that all forces are constant, whereas they are rarely so in Nature. Does not this often give an air of unreality to our mechanics work ? I know that without using calculus methods we cannot deal analytically with variable forces, but graphical methods are now in common use and present no great difficulties to young pupils. Thus we use graphical integration in the form of an area under a curve, *e.g.* the area under the velocity-time curve as giving the displacement. Why, then, pretend that in a blow the force of resistance is constant, when any thoughtful boy or girl knows that it need not be, and probably never is ? Having established the impulse-equation, we can proceed on Greenhill's lines as illustrated in Figs. 1-3. (The peak of the curve should be higher than that shown in Fig. (iii), and the time interval smaller.)

The whole difficulty over the observation and the concept of a blow or an impact is that we are not able, beforehand, to dose ourselves and our class with the "new accelerator" of Mr. H. G. Wells' story. So things move too fast for us ; the billiard balls separate before we are able to trace the changes in the force and the time. But this remark from Thomson and Tait may be illuminating. "If two spheres of about the size of the earth and made of some such material as glass or silver were to collide, the time of compression and restitution would be about an hour. For spheres of the same material of a yard in diameter, the time is about a thousandth of a second". I shall revert to this point later.

I quoted just now a remark of the late Sir George Greenhill. I do not imagine that Greenhill's original ideas on the teaching of Mechanics are very generally known. They are to be found in full in that absorbingly interesting but very diffuse *Notes on Dynamics*, published by the Stationery Office in 1908 and designed by the author for the advanced class of the Ordnance College, Woolwich. He also contributed a short paper on Linear Dynamics to the *Mathematical Gazette*, October 1914, and expanded this into a series of most entertaining articles called "Collision Dynamics" which appeared in *Engineering*, April and May 1920.

His main ideas may be summarised thus :

(i) Do not fuss about lecture-room experiments. They are artificial and on the wrong scale. Rather illustrate mechanical terms and principles from the events of everyday life—"the mega-micro Physics of Nature"—such as the motion of a car or a train, of the shot up the bore of a gun, and of the piston in a cylinder ; by a hammer striking a nail, the recoil of a gun, and pile-driving.

(ii) Do not, in the early stages, bother about logic or historical

order. We advance by being ignorant of our own ignorance, not by concentration on assumed knowledge. Start by familiarising the pupil with mechanical conceptions and processes. The logical framework will come later.

(iii) Shun absolute units as you would the devil.

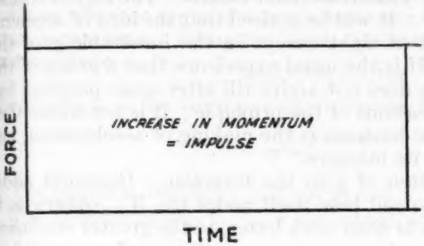

FIG. 1.

(i) The push.

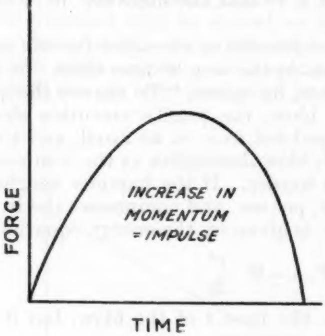

FIG. 2.

(ii) The shove.

FIG. 3.

(iii) The blow.

Thus he starts straight away with the energy and momentum equations :

A force F, lbs. or tons, acting for t seconds through s feet on a body of weight W lb. or tons (a railway train, tramcar, motor-bus, bicycle, boat, steamer, shot in a gun . . .) will give it a velocity v ft./sec. from rest, such that :

$$\begin{array}{ccccc} \text{lb.} & & & \text{lb.} & \\ \text{tons} & & \text{sec.} & \text{tons} & \text{sec.} \\ F & \times & t & = W \,. & \dfrac{v}{g} \end{array} \quad \text{momentum in lb.- or tons-seconds ;}$$

$$\begin{array}{ccccc} \text{lb.} & & & \text{lb.} & \\ \text{tons} & & \text{ft.} & \text{tons} & \text{ft.} \\ F & \times & s & = W \,. & \dfrac{v^2}{2g} \end{array} \quad \text{energy in lb.- or tons-feet.}$$

In a field of gravity g, where $F=W$, v is acquired in falling for $\frac{v}{g}$ seconds through $\frac{v^2}{2g}$ feet.

Notice that these two formulae are given without any justification. They are treated as formulae from an engineer's note-book. Their justification is that they yield reasonable results. The logical framework will be added later. It will be noticed that the idea of acceleration has been kept out of sight except in the inevitable g of the gravitation system. " It is the usual experience that a grasp of the meaning of acceleration does not arrive till after some progress has been made in the applications of the principle ; it is notorious that the engine-driver, whose business is the making of acceleration, has a very vague idea as to its measure."

Notice also the position of g in the formulae. Greenhill adds, " See that g does not go and hide itself under the W ; otherwise it will get absorbed into m or some such letter, to the greater confusion of everyone ". $\frac{v}{g}$ is the measure of a time and $\frac{v^2}{2g}$ is the same measure as the displacement s, so that the units sec.-lb., and ft.-lb. are justified.

The interest in Greenhill's elementary examples lies not so much in the examples themselves as in the way he uses them " to point a moral or adorn a tale ". Thus, he writes, " To answer the question of the static pressure of a blow, the pupil's attention should be directed to a smith forging red-hot iron on an anvil, and to noting how the compression at each blow diminishes as the iron cools, and the hammer seems to strike harder. If the hammer weighs W lb. and strikes with velocity v ft. per sec. and compresses the iron x ft., the average static force F lb. is given by the energy equation

$$F\,.\,x=W\,.\,\frac{v^2}{2g}.$$

It is not easy to measure the time t of the blow, but it can be inferred from the momentum equation

$$F\,.\,t=W\,.\,\frac{v}{g}.$$

Whence* it follows that $\quad t=x/\tfrac{1}{2}v.$

If the compression is $\frac{1}{4}$ in. for $v=20$ ft. per sec., t will be about $\frac{1}{480}$ of a second.

The blow of a hammer on soft iron could be studied formerly in the manufacture of the old muzzle-loading ' Woolwich Infant ', a gun made of a steel tube strengthened by a jacket of wrought-iron wire. When first removed from the furnace, the compression of the red-hot core made by the first few blows of the steam-hammer was considerable and the blow appeared soft ; as the iron cooled, the compression became smaller and the blow harder, or more violent in shaking the earth."

* The assumption here is that the space-average and the time-average of the force are equal.

Here is one of Greenhill's examples which he does not work out. I am using it to illustrate a point made just now.

A train, length a, reaches an incline, angle α. The tractive force is such that the train will run up the incline with constant speed. The speed is V when it reaches the foot of the incline. Find the time before the end of the train is on the incline.

Using absolute units, if m is the mass per unit length, the tractive force is $mga \sin \alpha$. Using the Newtonian method, the Second Law of motion gives

$$ma\ddot{x} = mg \sin \alpha (a - x),$$

the length x of the train having moved on to the incline. Put

$$n^2 = \frac{g \sin \alpha}{a}.$$

The equation becomes $\quad \ddot{x} = n^2(a - x).$

The solution of this is $\quad x = a - A \sin(nt + \epsilon).$

All the mechanics has now gone out of the question. It is now a matter of using initial conditions, viz., $x = 0$ and $\dot{x} = V$ when $t = 0$. A and ϵ can thus be determined.

Or the question may be solved on the lines of Huyghens. The resulting force on the train follows a linear law, decreasing from $mga \sin \alpha$ to $mg(a - x) \sin \alpha$, while s changes from 0 to x.

The work done is therefore given by the area of a trapezium.

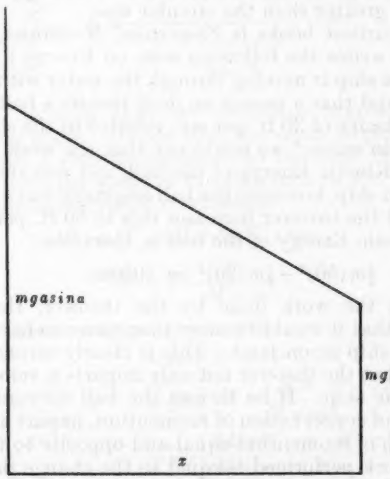

FIG. 4.

The energy equation is therefore

$$\tfrac{1}{2}ma(\dot{x}^2 - V^2) = \tfrac{1}{2}xmg \sin \alpha (2a - x),$$

or

$$\dot{x}^2 = V^2 + n^2a^2 - n^2(a - x)^2.$$

Put $$c^2 = V^2 + n^2a^2.$$

We have $$\frac{dx}{dt} = \sqrt{\{c^2 - n^2(a-x)^2\}}$$

and, therefore,

$$t = \frac{1}{n}\int_0^a \frac{dx}{\sqrt{\left\{\frac{c^2}{n^2} - (a-x)^2\right\}}}.$$

This integral is of a standard form, an inverse sine, and the answer comes out in a couple of lines to be

$$\frac{1}{n} \text{ arc sin } \frac{na}{\sqrt{\{V^2 + n^2a^2\}}}.$$

But that is not quite all. Greenhill now asks for the time taken for the train to run completely off the incline, on to the horizontal again.

The work is similar to the above, but there is a change of sign. The final result is $\frac{1}{n} \text{ arg sinh } \frac{na}{\sqrt{\{V^2 + n^2a^2\}}}$, *i.e.* the same as the former, except that the inverse circular function is replaced by an inverse hyperbolic one. The time of running off will be less than that of running on, since the steady velocity up the incline is greater than V (actually $\sqrt{\{V^2 + n^2a^2\}}$). This reminds us that the hyperbolic sine of an argument is greater than the circular sine.

One of Jeans' earliest books is *Theoretical Mechanics*, published in 1907. In it he writes the following note on Energy :

" Suppose that a ship is moving through the water with a velocity of 20 ft. per sec., and that a person on deck throws a ball of mass m forward with a velocity of 30 ft. per sec. relative to the ship. If the person were fixed in space,* we might say that the work he did was equal to the final Kinetic Energy of the ball, and was thus $\frac{1}{2}m(30)^2$, or $450m$. On board ship, however, the ball originally had a velocity of 20 ft. per sec., and the thrower increases this to 50 ft. per sec. The change in the Kinetic Energy of the ball is, therefore,

$$\tfrac{1}{2}m(50)^2 - \tfrac{1}{2}m(20)^2 \text{ or } 1050m.$$

If this represents the work done by the thrower, then we are driven to suppose that it would be more than twice as hard to throw the ball on board ship as on land. This is clearly erroneous. The error lies in this, that the thrower not only imparts a velocity to the ball but also to the ship. If he throws the ball forward he must, from the principle of conservation of momentum, impart a backward velocity to the ship of momentum equal and opposite to that of the ball. The total work performed is equal to the change produced in the total Kinetic Energy of the ship and the ball."

Jeans does not work out this problem in detail. I propose to do so, as it has many points of interest. First of all, let us look at it from another point of view. Suppose that in throwing the ball the

* Sir James Jeans would word this phrase differently in 1934!

man moves forward his arm through a horizontal distance of 3 ft. and that the average force he imparts is F. On shore, the work he does is $3F$. If, on board ship, he is carried forwards 4 ft. during the act of throwing, he imparts to the ball kinetic energy to the amount of $7F$. Though it is correct to say that he imparts this amount of kinetic energy to the ball, he does not supply it all from his store of internal energy—from his muscles and his meals. The extra kinetic energy comes from the ship, which loses that amount. It is transferred from the ship to the ball through the medium of the man's arm. But this curious point emerges from the analysis of the problem. On shore, to give the ball a velocity of 20 ft. per sec., $3F$ units of work are required. If it requires $7F$ units to give the ball the same velocity relative to the ship, less than $3F$ units have to be provided by the man and more than $4F$ units are transferred from the ship. The analysis follows.

A projectile is fired right ahead from a ship. The mass of ship is M (including gun, etc.); the mass of the projectile, m. The ship is moving with steady speed V through the frictionless water. (No currents, etc.) The explosive charge (or spring, or other means of

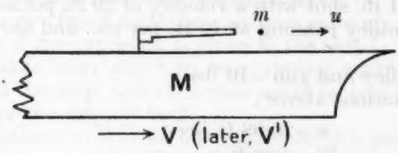

FIG. 5.

propulsion) is such that the projectile leaves the muzzle with velocity v on shore. On the ship, the same propulsive force gives to the projectile a velocity u relative to the ship. When the projectile leaves the gun, the speed of the ship falls to V'.

By the conservation of momentum:

$$(M+m)V = (M+m)V' + mu;$$

thus

$$V' = V - \frac{m}{M+m}.u.$$

Gain of kinetic energy by projectile

$$= \tfrac{1}{2}m[(V'+u)^2 - V^2]$$

$$= \tfrac{1}{2}m\left[\left(V + \frac{Mu}{M+m}\right)^2 - V^2\right]$$

$$= \tfrac{1}{2}m\left[\frac{2MuV}{M+m} + \left(\frac{M}{M+m}\right)^2 u^2\right].$$

Loss of kinetic energy by ship

$$= \tfrac{1}{2}M(V^2 - V'^2)$$

$$= \tfrac{1}{2}M\left[\frac{2muV}{M+m} - \left(\frac{m}{M+m}\right)^2 u^2\right].$$

Thus gain of kinetic energy by projectile minus loss of kinetic energy by ship

$$=\tfrac{1}{2}Mmu^2\left[\frac{M+m}{(M+m)^2}\right]=\tfrac{1}{2}mu^2\cdot\frac{M}{m+M}.$$

This is the energy provided by the explosion in the gun. If the gun be fired with the same explosive charge on shore, the velocity of the projectile is v.

Thus $$\tfrac{1}{2}mv^2=\tfrac{1}{2}mu^2\cdot\frac{M}{m+M},$$

or $$u=v\sqrt{\left\{1+\frac{m}{M}\right\}}.$$

So, to get an assigned value of u, we must *decrease* the force of propulsion from that required to give the same value of u on shore.

Note that gain of kinetic energy by projectile minus loss of kinetic energy by ship is independent of V.

Special cases.

(i) When M and m are comparable.

A gun fires a 1 lb. shot with a velocity of 20 ft. per sec. The gun is placed on a trolley running at 10 ft. per sec. and the gun is fired right ahead.

Weight of trolley and gun = 10 lbs.

Using the equations above:

$$u=20{\cdot}98\text{ ft./sec.}$$
$$V'=8{\cdot}09\text{ ft./sec.}$$
$$u+V'=29{\cdot}07\text{ ft./sec.}$$

Kinetic energy of shot is increased by 373 ft. poundals.

200 ft. poundals of this are provided by the explosion, 173 ft. poundals are transmitted from the trolley, which loses this amount.

(ii) When m is small compared with M (*e.g.* a small gun on H.M.S. *Nelson*),

$$V'=V-\delta V \text{ where } \delta V=\frac{mu}{m+M},$$
$$V^2-V'^2=2V\,\delta V.$$

Loss of kinetic energy by ship $=\tfrac{1}{2}M(V^2-V'^2)$

$$=\tfrac{1}{2}\frac{mM}{m+M}\cdot 2uV$$

$=muV$ to a first approximation.

Gain of kinetic energy by projectile $=muV+\tfrac{1}{2}mu^2$.

To the approximation considered $u=v$.

If $M=30{,}000$ tons, $V=25$ ft.-sec. (about 15 knots),
$m=300$ lbs., $v=600$ ft./sec.

Loss of kinetic energy by ship $=4{\cdot}5\times10^6$ ft.-poundals.
Explosion gives 54×10^6 ft.-poundals.

Gain of kinetic energy by shell $=58\cdot5\times10^6$ ft.-poundals.

Note that in this case $V-V'=0\cdot0027$ ft.-sec. and $\frac{1}{2}(V^2-V'^2)=0\cdot067$.

To get the loss of kinetic energy of the ship, this is multiplied by M, which is very large, so yielding a measurable result.

[Compare with the impulse of a blow, $F(t-t')$, when F is large and $t-t'$ small.]

(iii) Suppose the gun is fired astern. Put $u=-u'$ in the previous work.

Gain of kinetic energy by projectile is

$$\tfrac{1}{2}m\left[\left(\frac{M}{M+m}\right)^2u'^2-\frac{2M}{M+m}u'V\right].$$

This is zero if $$u'=\frac{M+m}{M}\cdot2V=2V+\frac{m}{M}\cdot2V.$$

Then $$V'=V+\frac{m}{M}\cdot2V.$$

The speed astern of the projectile relative to the earth is $u'-V'$ or V, as is obvious from other considerations.

There we are faced with the apparent anomaly that the whole of the energy of the explosion is given to the ship! The energy of the projectile falls from $\frac{1}{2}mV^2$ to zero, during which time it is being given up to the ship, and then rises again to $\frac{1}{2}mV^2$, taking from the explosion or the ship, or both.

As an analogy to this we have the case of a perfectly elastic ball bounced against a wall. There is no change in kinetic energy, equal amounts of work having been done by the ball on the wall and by the wall on the ball. But there is a change of momentum from mV to $-mV$.

What it comes to is that an expression for energy-change is not, by itself, a complete description of a dynamical change. A momentum expression is needed as well.

III.

Finally, let us consider again the problem of Direct Impact. First I will indicate the lines of Greenhill's work, adapted from Maxwell's *Matter and Motion*. Then I will go on to the analysis of an experiment, devised and carried out by Mr. E. W. Kempson, Headmaster of the Royal Naval College, Dartmouth, and briefly described in the M.A. *Report on the Teaching of Mechanics*. Two trolleys A and B are placed on a plane, slightly inclined to counteract the effects of friction, so that each will run, under no force, with uniform speed. In the particular experiment, they were of equal mass. B is ahead of A. A carries at its forward end a large watch-spring, of about a foot in diameter. B is initially at rest, and A is set in gentle motion. As soon as the forward end of the spring reaches B, B starts to move and A to slow up.

At the moment of maximum compression of the spring the speeds of the trolleys are equal, the sum of their kinetic energies being one-

half of the original kinetic energy of A. The other half is stored up in the spring as potential energy. After this the spring loses its compression. Finally, A comes to rest and B moves forward with the original speed of A. The elasticity of the spring is as near "perfect" as it is possible to get in practice. The value of the experiment lies primarily in the fact that the operation is slow enough to follow with the eye. In a particular trial, of which I have the results, the whole time taken was a little over a second. It represents something between a push and a blow (what I called just now, a shove). But, with care, quantitative results may be obtained. A spring of frequency 10 was used, carrying the usual inked brush. From this, the trace for each trolley could be obtained. Hence the displacement-time and velocity-time graphs could be drawn. Mr. Kempson wished to compare his experimental results with those given by analysis. For this purpose it was desirable to choose something which did not depend on the constant of the spring. He measured, therefore, the ratio of the distance travelled by A up to the instant of maximum compression, to the total distance run during compression and restitution. I append the mathematics. It is quite ordinary differential equation work. The interest lies in noting the closeness of the approximation of the experimental to the theoretical results.

Direct Impact. The method of Sir George Greenhill (see *Gazette*, Dec. 1914).

Consider two trucks, trolleys . . . moving in the same straight line. Weights, w_1, w_2. Speeds, u_1, u_2.

The combined kinetic energy may be written $A + B$,

where
$$A = (w_1 + w_2)\frac{U^2}{2g},$$
$$B = \frac{w_1 w_2}{w_1 + w_2} \cdot \frac{(u_1 - u_2)^2}{2g}.$$

U being $\dfrac{w_1 u_1 + w_2 u_2}{w_1 + w_2}$, the velocity of the centre of gravity of the trucks.

A is called the molar energy, being the kinetic energy of the whole weight concentrated at the centre of gravity.

B can be written $w_1 \cdot \dfrac{(u_1 - U)^2}{2g} + w_2 \cdot \dfrac{(U - u_2)^2}{2g}$.

This is the sum of the kinetic energies relative to the centre of gravity. It is called the internal molecular energy.*

When there is inelastic collision, B is dissipated or liberated.

When the bodies are imperfectly elastic, $(1 - e^2)B$ is dissipated or liberated.

In this case, the speed of each body relative to the centre of gravity is reduced, after collision, in the ratio $e : 1$.

In the case of perfect resilience, $e = 1$. Assume the "law of the

* For a geometrical method of reaching these results, see *Gazette*, Dec. 1933, note by C. G. Paradine.

spring". If the thrust is F lbs. when the spring is compressed a maximum amount a,

$$\tfrac{1}{2}F \cdot a = \frac{w_1 w_2}{w_1 + w_2} \cdot \frac{(u_1 - u_2)^2}{2g}.$$

The impact during compression lasts $\dfrac{\pi a}{2(u_1 - u_2)}$, and the rebound lasts the same time.

So the impact, during compression and restitution, lasts

$$\frac{\pi a}{u_1 - u_2} = \pi \sqrt{\frac{a}{2g}} \sqrt{\left\{\frac{w_1 w_2}{F(w_1 + w_2)}\right\}}.$$

This is constant, since a/F is constant (law of the spring). Thus the impact lasts the same time, whatever be the velocity of approach $u_1 - u_2$.

Experiment on Direct Impact. (See M.A. *Report on the Teaching of Mechanics*, p. 67.)

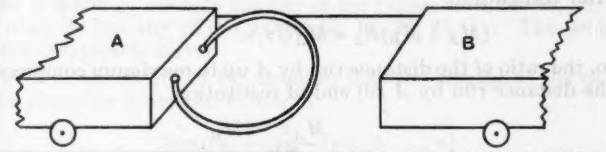

FIG. 6.

A, B are trolleys of masses M_A and M_B. The plane is sloped towards B to eliminate friction. B is initially at rest. A moves with initial speed U and collides with B.

Measure distances from O, the forward end of A when the spring makes first contact with B.

The diameter of the spring is b.

FIG. 7.

At a subsequent instant t, S_A and S_B are the distances of the bows of A and the stern of B from O. v_A and v_B are the speeds.

$$M_A v_A + M_B v_B = M_A U \quad \text{(conservation of momentum)}.$$

Therefore, by integration,

$$M_A S_A + M_B S_B = M_A U t + M_B b. \quad \text{.....................(i)}$$

The compression of the spring, y, is $b - (S_B - S_A)$.

The "law of the spring" gives $\ddot{y} = -\mu^2 y$ (μ depends on the constant of the spring, M_A, and M_B).

Therefore
$$y = C \sin \mu t + D \cos \mu t.$$

Now $D = 0$, since $y = 0$ when $t = 0$.

Thus
$$b - (S_B - S_A) = C \sin \mu t.$$

Differentiating, $\quad -v_B + v_A = C\mu \cos \mu t.$

When $t=0$, $v_A = U$, $v_B = 0$, and so $C = U/\mu$.

Whence we have,

$$S_B - S_A = b - \frac{U}{\mu}\sin \mu t. \quad \text{.......................(ii)}$$

From (i) and (ii),

$$(M_A + M_B)S_A = M_A U t + \frac{M_B U \sin \mu t}{\mu}.$$

When the spring is totally compressed, $\mu t = \frac{\pi}{2}$.

Hence at this instant,

$$(M_A + M_B)S_A = \frac{M_A U \pi}{2\mu} + \frac{M_B U}{\mu}.$$

After restitution,

$$(M_A + M_B)S_A = M_A U \pi/\mu.$$

So, the ratio of the distance run by A up to maximum compression to the distance run by A till end of restitution

$$= \frac{M_A \pi + 2M_B}{2M_A \pi}.$$

If $M_A = M_B$, this is $\frac{1}{2} + \frac{1}{\pi} = 0\cdot 818 \ldots$.

An experiment at Dartmouth gave: distance run up to maximum compression, 33 cm.; total distance run, 41 cm. Ratio $\frac{33}{41} = 0\cdot 806 \ldots$.

C is the maximum compression of the spring.

We had $\mu = \frac{U}{C}$. Therefore, total time of impact $= \frac{\pi}{\mu} = \frac{\pi C}{U}$, in agreement with Greenhill's result. The form $\frac{\pi}{\mu}$ shows it to be independent of U.

H. E. P.

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986. The volume of the waters of the world is infinitely greater than its land surface. . . .—Basil de Selincourt in the *Observer*, 9th July, 1933. [Per Dr. W. G. Bickley.]

987. Mathematics should be taught, he urges, by a highly-paid professional dunce, since " no mathematician has any idea of the difficulties which children have over the elements of the subject ".—Account of a lecture by Dr. Reginald Miller given in *The Observer*, 21st May, 1933. [Per Mr. H. V. Lowry.]

988. And when, during the month of March, our own gardens gay with snowdrops and crocuses, we switched on to Moscow, we heard something like this : " The weather is beginning to get much milder. The temperature to-day is only 10 deg. below zero ". That came through in English, and to make quite sure of it, the announcer is careful to point out that those 10 deg. below zero are degrees Centigrade, leaving our old pal Fahrenheit beaten a distance.—Peter Brown, in the *Sunday Pictorial*, 16th July, 1933. [Per Prof. E. H. Neville.]

A PROBLEM OF DISTRIBUTION.

By G. N. Watson.

The following problem was proposed to me recently by its inventor, Mr. G. E. Crawford,* who had not got a solution of it, except for some small values of n; the solution which I have constructed seems to me to be not so obvious that it is not worth putting on record.

On the circumference of a circle $2n$ points are marked so as to form the vertices of a regular polygon of $2n$ sides inscribed in the circle. It is required to ascertain whether it is possible to draw a system of n chords of the circle such that each of the marked points is an end of one of the chords and such that no two chords are of the same length; and, in the event of at least one such system existing, it is required to construct a system satisfying the two conditions stated.

I first prove that the problem has no solution when $2n$ has any of the values 4, 6, 12, 14, 20, 22, 28, 30, ..., and I then state solutions of the problem (i) when $2n$ has any of the values 8, 16, 24, 32, ..., and (ii) when $2n$ has any of the values 10, 18, 26, 34, ... The problem is thus completely solved.

Let a be the radius of the circle so that the lengths of the chords to be placed in it are the values of

$$2a \sin \frac{p\pi}{2n}; \quad (p = 1, 2, 3, \dots, n)$$

let these chords be numbered $1, 2, 3, \dots, p$ in ascending order of magnitude.

Also number the points marked on the circle $0, 1, 2, \dots, 2n-1$ in order counterclockwise.

Take the minor arc of the circle cut off by the chord and let the numbers of its end-points be c_p and d_p, it being supposed that c_p is the first point of the arc and d_p the last point when the arc is traversed counterclockwise.

Then $$d_p - c_p = p \text{ or } p - 2n.$$

When we sum results of this type we get

$$\sum_{p=1}^{n} d_p - \sum_{p=1}^{n} c_p \equiv \sum_{p=1}^{n} p, \quad (\text{mod } 2n)$$

so that $$\sum_{p=1}^{n} c_p + \sum_{p=1}^{n} d_p - \sum_{p=1}^{n} p \equiv 2 \sum_{p=1}^{n} c_p. \quad (\text{mod } 2n)$$

Now the numbers c_p, d_p $(p = 1, 2, 3, \dots, n)$ are the numbers $0, 1, 2, \dots, 2n-1$ in some order, and hence we have

$$\frac{2n(2n-1)}{2} - \frac{n(n+1)}{2} \equiv 2 \sum_{p=1}^{n} c_p, \quad (\text{mod } 2n)$$

that is to say $$\tfrac{3}{2} n(n-1) \equiv 2 \sum_{p=1}^{n} c_p. \quad (\text{mod } 2n)$$

* My thanks are due to Mr. Crawford for calling my attention to the problem, and for the pleasure which I derived from attacking a problem outside my usual routine.

Hence $\frac{3}{2}n(n-1)$ must be an even number, and so n must be either of the form $4m$ or of the form $4m+1$. This proves that the problem has no solution when $2n$ has any of the values 4, 6, 12, 14, 20, 22, 28, 30, ...

A solution of the problem when n is of the form $4m$ and $m \geqslant 2$ is given by the following Table I, in which the column headed p gives the number of each chord while the columns headed c_p and d_p give the numbers of the marked points at the ends of the chords. By enumerating the groups of marked points in the order of the roman numerals appended to them, we see immediately that each marked point occurs once and only once. The even values of p are given in the first part of the table and the odd values in the second part.

TABLE I.

	c_p	p	d_p	
	$4m-1$	2	$4m+1$	
	$4m-2$	4	$4m+2$	
(vi)	...	...	...	(viii)
	...	...	...	
	$2m+1$	$4m-2$	$6m-1$	
(i)	0	$4m$	$4m$	(vii)
(iii)	$m-1$	1	m	(iv)
	$8m-2$	3	1	
	$8m-3$	5	2	
(xi)	...	...	...	(ii)
	...	...	...	
	$7m+1$	$2m-3$	$m-2$	

	c_p	p	d_p	
(ix)	$6m$	$2m-1$	$8m-1$	(xii)
	$7m$	$2m+1$	$m+1$	
	$7m-1$	$2m+3$	$m+2$	
(x)	...	...	...	(v)
	...	...	...	
	$6m+1$	$4m-1$	$2m$	

The corresponding table (Table II), when n is of the form $4m+1$ and $m \geqslant 2$, is as follows :

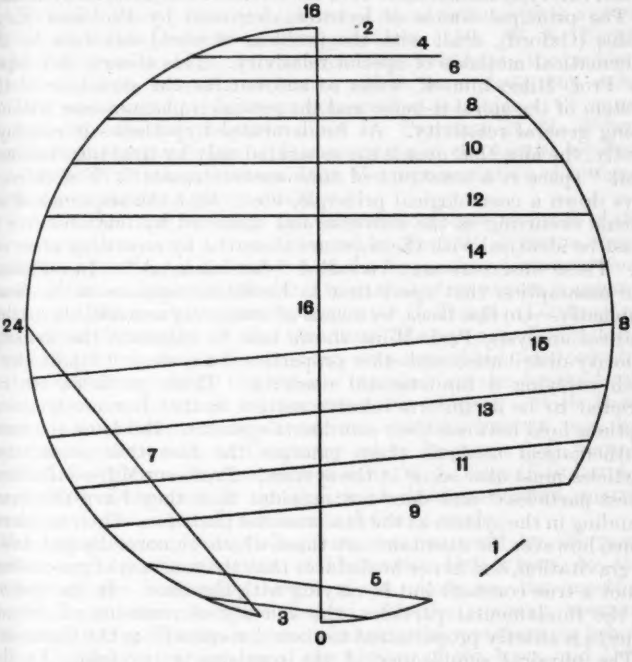
TABLE II.

	c_p	p	d_p	
	$4m$	2	$4m+2$	
	$4m-1$	4	$4m+3$	
(vi)	...	...	...	(viii)
	...	...	...	
	$2m+1$	$4m$	$6m+1$	
(iii)	$m-1$	1	m	(iv)
	$8m$	3	1	
	$8m-1$	5	2	
(xi)	...	...	...	(ii)
	...	...	...	
	$7m+3$	$2m-3$	$m-2$	

	c_p	p	d_p	
(ix)	$6m+2$	$2m-1$	$8m+1$	(xii)
	$7m+2$	$2m+1$	$m+1$	
	$7m+1$	$2m+3$	$m+2$	
(x)	...	...	...	(v)
	...	...	...	
	$6m+3$	$4m-1$	$2m$	
(i)	0	$4m+1$	$4m+1$	(vii)

The groups numbered (ii) and (xi) do not occur in these Tables when $2n$ is equal to 16 or 18.

When $2n$ is 8 or 10, these solutions break down, but it is an easy matter to construct solutions * in these special cases. Thus, when $2n=8$, we may take the ends of the chords to be the following pairs of marked points: (0, 1), (3, 5), (4, 7), (2, 6). When $2n=10$, we may take the ends of the chords to be the following pairs of marked points: (0, 1), (3, 5), (6, 9), (4, 8), (2, 7); or else: (0, 1), (4, 6), (5, 8), (3, 9), (2, 7).



To illustrate the solution in the general case, I give a diagram for $2n=32$. For the sake of clearness I have omitted the numbers of all of the marked points other than the four cardinal points, though the numbers of all the chords are inserted. The reader will observe three characteristic features of my solution in the general case, namely the vertical diameter, and the disposal of the chords numbered 1 and (for $2n=32$) 7. G. N. W.

* The solutions stated are the only fundamental solutions for $2n=8$ and $2n=10$.

EDINBURGH MATHEMATICAL SOCIETY: ST. ANDREWS COLLOQUIUM.

By G. C. McVittie.

The Colloquium held at St. Andrews from July 18 to 28 by the Edinburgh Mathematical Society was the third such meeting held since the War and proved as successful as on previous occasions. Some sixty-five persons attended, including fourteen professors of mathematics drawn from the Universities of England, Scotland, Ireland and Egypt and also Professor W. de Sitter, the Director of the Observatory at Leiden and one of the foremost astronomers of the day.

The principal course of lectures, delivered by Professor E. A. Milne (Oxford), dealt with the problem of world-structure by the kinematical methods of special relativity. This theory, developed by Prof. Milne himself, seeks to account for the structure of the system of the spiral nebulae and the recession phenomenon without using general relativity. As fundamental hypotheses it employs, firstly, the idea that events are separated only by time intervals and that "space is a construct of time measurements". Secondly, it lays down a cosmological principle, viz.: that the sequence of all events occurring in the universe and observed by one observer A must be identical with the sequence observed by any other observer B. These observers may be called "fundamental". In addition, the assumption that space-time is Euclidean appears to be made implicitly. On this basis, by means of some very remarkable mathematical analysis, Prof. Milne shows how to calculate the density, velocity-distribution and other properties of a system of "particles" each carrying a fundamental observer. These particles are restricted to be in uniform relative motion so that Lorentz transformations hold between their coordinate-systems. But from the same mathematical analysis there emerges the fact that accelerated particles must also occur in the system. Professor Milne calls these "test-particles" and does not consider that they have the same standing in the system as the fundamental particles. Their accelerations, however, he maintains are those which we normally put down to gravitation, and hence he deduces that the constant of gravitation is not a true constant but is varying with the time. In the system of the fundamental particles, the velocity of recession of distant objects is strictly proportional to their distances from the observer.

The physical significance of the particles is two-fold. In the earlier part of the theory they can be identified with spiral nebulae, but in later refinements they are supposed to be some more fundamental type of particle out of which the nuclei of spiral nebulae are formed and which also give rise to cosmic rays. For one of the consequences of the theory is that agglomerations of particles must be formed in time (thus forming the nuclei of the nebulae) whilst there must also be particles which attain enormously high velocities. These, on colliding with slower-moving groups of particles, would give rise to cosmic rays.

In a discussion on these lectures led by Dr. W. H. McCrea (London)

in which Prof. W. de Sitter and others took part, it was pointed out that Prof. Milne's theory had brought into prominence the fact that in general relativity also the scattering of the nebulae went on independently of gravitation. But the exact relation between the two theories had yet to be worked out. It was also not made quite clear in Prof. Milne's theory how much of the results depended on the initial assumption that space-time was Euclidean and how much was really independent of the geometry.

In the field of pure mathematics, Professor H. W. Turnbull (St. Andrews) spoke on Pictorial Geometry, dealing with such questions as the generalised construction for the ellipse and hyperbola, the densest and loosest packing of spheres of equal radius within a given volume and the problem of configurations. Mr. W. L. Ferrar (Oxford) gave an account of some expansions relating to the problem of lattice points, treating of lattice points in a circle, the order problem and relations between summation formulae. Of equal interest were the two lectures given by Professor B. M. Wilson (Dundee) on the notebooks of Ramanujan and the lecture by Professor J. M. Whittaker (Liverpool) on the representation of integral functions by series of polynomials.

Of the less formal meetings special mention must be made of the two evening discourses. One, delivered by Prof. W. de Sitter, was a masterly exposition of the subject of the Expanding Universe from the observational point of view and from that of an expert in general relativity. The second, given by Professor G. Temple (London), on the General Principles of the Quantum Theory and Eddington's theory of the fine-structure constant, was remarkable not only for the breadth of knowledge the lecturer revealed but also for the fascinating manner in which he presented his subject. Both these lectures aroused interesting discussions. A like interest was displayed in the discussion on Geometry led by Professor J. G. Semple (Belfast), Dr. Timms (St. Andrews) and Mr. W. L. Edge (Edinburgh), who, taking a theorem in the theory of three associated quartic curves, each gave a proof of it from a different angle.

The lighter side of the Colloquium was favoured by the good weather and by the situation of University Hall where, by the courtesy of the University Court of St. Andrews, the meeting was held and the members housed. At the opening meeting Professor D'Arcy Thompson, on welcoming the members on behalf of the University, spoke also of the history of the town and University. The members were indebted to Professor and Mrs. Turnbull who held a reception to which a number of the residents of St. Andrews were also invited. In the afternoons golf was naturally a great attraction, but time was found for two tennis tournaments and an excursion to Loch Earn, which were all much enjoyed by those taking part. Equally successful were two informal concerts organised by the members. The evident keenness of those present for all aspects of the Colloquium encourages the Committee of the Society to believe that these meetings perform a useful function and should be continued in the future. G. C. McV.

THE ORGANISATION AND INTERRELATION OF SCHOOLS.

MATHEMATICS.

MEMORANDUM FROM THE MATHEMATICAL ASSOCIATION.

THE members of the Sub-Committee appointed by the General Teaching Committee of the Mathematical Association to prepare a memorandum on the Organisation and Interrelation of Schools for the Board of Education Consultative Committee wish their signatures to convey general agreement with the conclusions set out in this memorandum though not necessarily their unanimous approval of every detail.

(*Signed*) J. H. BURDON. E. M. READ.
M. HAMMOND. A. W. RILEY.
B. A. HOWARD. W. J. LANGFORD.

1. *Scope of the Memorandum.*

1·1. The Sub-Committee appointed by the Mathematical Association to prepare this memorandum for the Consultative Committee of the Board of Education confined itself to a review of the situation as it affects, and is affected by, the study of Mathematics and Mechanics.

1·2. In the absence of any statement to the contrary this memorandum must be read as referring only to the schools covered by the terms of reference, and to the study of mathematics in those schools.

2. *School Organisation.*

2·11. Secondary Schools.

Under the present system most schools enter as many pupils as possible for a School Certificate examination. This inevitably leads to a standardised syllabus, and little regard can be paid to the *future* needs of the individual. It is felt that this present system is completely satisfactory only for that class of pupils who will proceed to a further education of University standard which includes mathematics as a main subject. For other classes the present position is not satisfactory and they are discussed as follows:

(*a*) Pupils who will proceed to a further education of a University type which does not include mathematics as a main subject.

(*b*) Pupils who will proceed to a further education of a technical character which includes mathematics as a main subject.

(*c*) Pupils who will follow some fairly extended commercial course such as that required in Banking or Insurance.

(*d*) Pupils who will proceed to a further education of a technical or commercial type in which mathematics is not a main subject.

(*e*) Pupils whose education is not continued beyond the age of 16+.

Each of these groups will, undoubtedly, contain some pupils able to reap full benefit from the present system, but it is felt, nevertheless, that some modification of the existing curriculum is desirable.

2·12. For pupils in (*a*) the course is too academic though often well within their capacity. It is felt that the mathematical curriculum should be broadened, preferably by a post-certificate course which would include a study of the history of mathematics and give some acquaintance with the practical applications of mathematics over as wide a field as possible. The whole object would be to indicate the place of mathematics in modern life and thought. Subject to the inclusion of some such course the present School Certificate curriculum could be regarded as satisfactory.

2·13. Many pupils included in (*b*) reach credit standard in School Certificate mathematics and yet experience considerable difficulty in adapting and applying the mathematics they have learned. In practice this often resolves itself into an irksome, if not wasteful, delay before they can embark upon the ultimate course, as for example the course for a National Certificate in Engineering. This delay would be avoided if the pupils could take up the practical part of the subject at an earlier stage. It would be worth while to sacrifice some of the more academic parts in the existing School Certificate course to make this possible.

2·14. For pupils in (*c*) much the same difficulties arise as in (*b*). Some of these pupils have reached a high standard in School Certificate mathematics and yet find that they are hampered in their studies owing to the fact that some applications of elementary work, necessary for their particular line of study, were not included in the School Certificate course. The omissions are often small in extent and could be included in a School Certificate course without undue strain if the situation were fully explored.

2·15. Many of the pupils in (*d*) and (*e*) never attain any real mastery over the branches of the subject they have studied. They are not, for the most part, lacking in intelligence in the broad sense, and yet many leave school at the age of 16 + without anything in the form of a certificate to show either the course of study they have followed or the standard reached in any subject; without any knowledge of mathematics likely to be of use to them in after life, but often with a distaste for the subject. Although such pupils are able to carry out rule-of-thumb calculations with reasonable ability and accuracy and may often excel in something related to mathematics, *e.g.*, accurate geometrical drawing (although it is a mistake to suppose this is generally true) they seem unable to deal with that part of the work requiring much logical argument. Such pupils are, as a rule, capable of good work in the more straightforward parts of Arithmetic, the bare elements of Algebra, Mensuration, practical Geometry and numerical Trigonometry. It is recommended that a full course of this nature should be allowed them.

2·16. For pupils in groups (*b*), (*d*) and (*e*) above, the solution to the problem would seem to be the reorganisation of Secondary

Schools somewhat on the lines of the dual bias selective Central Schools. The division, *within the school*, should become operative from about the age of 13 + and would ensure a course of study more suited to the ability and requirements of the pupils.

2·17. The Board of Education Report on the Education of the Adolescent advocates the transfer from Secondary Schools to Central Schools or Junior Technical Schools of those pupils who, having begun their career in a Secondary School, later show signs that they are better adapted to a less academic type of education. Such a procedure, carried out with the co-operation of those in charge of both types of school, would go far to solve the problems created by the pupils in groups (*d*) and (*e*). Against this must be urged the disadvantages which might ensue a change of school half-way through the post-primary school career.

2·2. Other Schools within the terms of reference.

The Mathematical Association has few members representing these schools, and is not closely in touch with the standard or scope of their work. As far as the evidence goes it appears to show that the study of mathematics is usually undertaken with ultimate applications in view. In most cases there is ample scope for the use of the subject matter and the results of elementary mathematics. There is, however, reason to believe that rules are often used with little or no reference to the underlying principles. While it is evident that a full discussion of theory would be out of place, if not impossible, some effort should be made, when opportunity occurs, to disclose principles and so prevent the use of rules from being regarded as a rule-of-thumb operation.

Subject Matter.

3·11. Secondary Schools.

Experiments are always to be encouraged although the existing state of affairs amended in accordance with 2·12, 2·13, 2·14 could be considered satisfactory for all pupils except those in groups (*d*) and (*e*). Whether the pupil leaves at 16 + or remains to take an advanced course of study the foundation of principles and processes should prove sufficient for later requirements. Published Reports * of the Mathematical Association deal (i) with the content and presentation of various branches of the subject and (ii) with the needs of various types of pupil.

3·12. For those pupils in groups (*d*) and (*e*) the Mathematical Association *Report on the Teaching of Mathematics to Evening Technical Students* covers some of the ground (see pp. 11, 12, 15, 16).

* (i)
- *The Teaching of Geometry in Schools*, 1923 (3rd Edn. 1929).
- *The Teaching of Mechanics in Schools*, 1930.
- *The Teaching of Arithmetic in Schools*, 1933.
- *The Teaching of Algebra in Schools*, 1934.

(ii)
- *The Teaching of Mathematics in Public and Secondary Schools*, 1928.
- *The Teaching of Elementary Mathematics in Girls' Schools*, 1933.

Much valuable help is also contained in the other Reports of the Mathematical Association and the Board of Education publications:

Practical Mathematics. Perry (1903).

Teaching of Mathematics in the United Kingdom, Part II. (1912), pp. 10-42.

These pupils require a comprehensive treatment of Arithmetic linked with some portions of elementary Algebra, Geometry, Trigonometry and Mechanics. Sufficient stress must be laid on the principles to ensure that the application is not entirely mechanical, while the selection of subject matter should be governed by its application to the social and physical environment. The report of a discussion on Mathematics in Central Schools (*Mathematical Gazette*, May 1934, pp. 80-94) gives the details of attempts which are being made to carry out a course such as that outlined above.

3·2. Other Schools.

In the other schools covered by the terms of reference the subject matter dealt with in mathematics is directly linked with the whole course of study. Subject to the observance of the suggestions in 2·2 the present position can be regarded as satisfactory.

4. *Teaching Staff.*

4·1. Co-operation of Teachers.

The writer of the article on the Preliminary Mathematical Training of Technical Students (Board of Education *Report on the Teaching of Mathematics in the United Kingdom*, Part II (1912), p. 25) puts forward a plea for conferences of teachers of mathematics in different types of schools, particularly those who deal with the different phases of the education of the same pupil. The value of such conferences to the teachers and ultimately to the pupils is undoubted. At present it appears that there is little co-ordination between the courses in Secondary Schools and those in Senior Technical and Commercial Schools.

4·2. Training of Teachers.

The teaching of mathematics to all types of pupil considered in this memorandum, with exception of those pupils in Secondary Schools who ultimately proceed to a further education of University standard, demands a training different from that which is usually available in the Universities and Training Colleges. It is true that the actual experience of teaching will show the teacher how far his academic study of mathematics has failed to fit him for the task of imparting the elements of the subject to the pupil. The training in the technique of teaching will help him to present what is of value in the most suitable way, but in dealing with pupils who will ultimately take some technical course of study it often happens that the teacher has had little or no acquaintance with the applications of mathematics which the pupil will meet in the future, as, for example, the use of duodecimals in the building trade. The teachers who will have the responsibility of preparing mathematical scholars

undoubtedly need all the mathematics that can be gained from a University career, but the same cannot be said of the teacher whose life's work is linked with the pupils whose needs are of a more technical character. It is recommended that lecturers in the teaching of mathematics should take steps to acquaint themselves with the applications of the subject to technical and commercial affairs, so that they may more adequately prepare their students to teach in schools other than those of the strictly academic type. The claims of the pupils who need their mathematics applied should be seriously considered, and it is clear that the best results cannot be obtained in the schools until the teacher has been prepared for the problems which have to be faced.

TEXT OF THE REFERENCE FROM THE BOARD OF EDUCATION.

"To consider and report upon the organisation and interrelation of schools, other than those administered under the Elementary Code, which provide education for pupils beyond the age of 11+; regard being had in particular to the framework and content of the education of pupils who do not remain at school beyond the age of about 16+."

---

989. Double Acrostic.

A mathematician's selections
Have no turf connections.

1. The unit of distance appears
To be $3\frac{1}{4}$ light years.
2. A sound reflection
3. Makes a light collection.
4. Greek success with roots was rare
Thymaridas produced this flower so fair.
5. Eleven hundred and − less,
A factor of the product guess.

6.
7. Lo here the "Cossic Art"
8. Doth T and T' part.

9. "Pocket Barrême"
(One scarce hears his name)
10. Not quite
A hidden light.
11. Burns went to Plough, Byron to Harrow,
This Lincoln farmer followed his Barrow.
12. Wise he doth appear
And truthful here.

Lights 3 and 4 are reversed.

SUR LE TRIANGLE ISOCÈLE.

PAR V. THÉBAULT.

M. N. ALTSHILLER-COURT vient de donner une bibliographie de ce petit problème célèbre * : " Tout triangle qui a deux bissectrices intérieures égales est isocèle ". En plus des solutions que nous avons données (*Mathesis*, 1930, p. 97 ; 1932, p. 162), voici trois autres démonstrations assez simples.

1. *Lemme.* Un quadrilatère convexe dans lequel deux côtés opposés et deux angles opposés *obtus* sont égaux est un parallélogramme.

Soit un triangle *quelconque* ABC. Les bissectrices intérieures BE, CF, des angles B, C, limitées aux côtés opposés AC, BA, se coupent en I. Construisons le triangle BDE comme l'indique la figure 1.

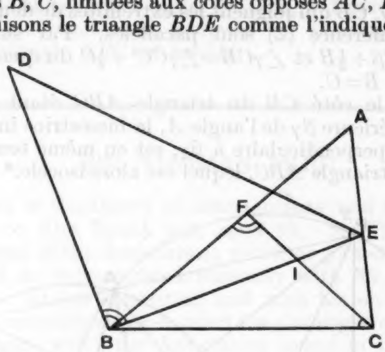


FIG. 1.

On a ces égalités

$$\angle BIC = \angle BEC + \angle ECF = \angle BEC + \tfrac{1}{2}C$$
$$= \angle BFC + \angle EBF = \angle BFC + \tfrac{1}{2}B,$$

d'où

$$\angle BIC = 90^\circ + \tfrac{1}{2}A = \angle DEC = \angle CBD.$$

Si on suppose que $BE = CF$, les triangles BFC, BDE sont égaux et le quadrilatère $BDEC$, qui a deux côtés égaux ($BC = DE$), puis deux angles opposés *obtus* et égaux ($\angle CBD = \angle DEC$), est un parallélogramme. Alors,

$$\tfrac{1}{2}C = \angle DEB = \angle EBC = \tfrac{1}{2}B.\dagger$$

2. Sur les côtés BA, AC d'un triangle *quelconque* ABC, on marque des points arbitraires C' et B'. Les cercles (w_b), (w_c), circonscrits aux triangles BAB' et $CC'A$, coupent respectivement la bissectrice extérieure de l'angle A en β, γ. Les triangles $B\beta B'$, $C'\gamma C$ sont isocèles et semblables, à cause des égalités d'angles :

$$\angle \beta B'B = \angle \beta AB = 90^\circ - \tfrac{1}{2}A = \angle CA\gamma = \angle B'B\beta,$$

* *Math. Gazette*, XVIII (May 1934), p. 120.

† V. Thébault, *Education mathématique*, XXXIV, p. 65.

$$\angle CC'\gamma = \angle CA\gamma = 90° - \tfrac{1}{2}A = \angle \beta AB = \angle \gamma CC',$$
$$\angle B\beta B' = \angle C'\gamma C = \angle BAC = A.$$

Supposons que les points B', C' coïncident respectivement avec les pieds des bissectrices intérieures des angles B et C. Les angles opposés $B\beta\gamma$ et γCB du quadrilatère $B\beta\gamma C$ s'expriment

$$\angle B\beta\gamma = \angle B\beta B' + \angle B'\beta A = \angle B\beta B' + \angle B'BA = A + \tfrac{1}{2}B,$$
$$\angle \gamma CB = \angle \gamma CC' + \angle C'CB = 90° - \tfrac{1}{2}A + \tfrac{1}{2}C,$$

et $\angle B\beta\gamma + \angle \gamma CB = 180°$.

Le quadrilatère $B\beta\gamma C$ est donc inscriptible dans une circonférence (Σ). Si $BB' = CC'$, les cercles (w_b), (w_c) sont égaux ainsi que les triangles isocèles $B\beta B'$, $C'\gamma C$. Les cordes $B\beta$, $C\gamma$ sont donc égales et les cordes $\beta\gamma$, CB qui joignent les extrémités de deux cordes égales dans la circonférence (Σ) sont parallèles. Par suite, les angles $\angle CB\beta = \angle B'B\beta + \tfrac{1}{2}B$ et $\angle \gamma CB = \angle \gamma CC' + \tfrac{1}{2}C$ du quadrilatère $B\beta\gamma C$ sont égaux, et $B = C$.

Autrement : le côté CB du triangle ABC étant parallèle à la bissectrice extérieure $\beta\gamma$ de l'angle A, la bissectrice intérieure de cet angle, qui est perpendiculaire à $\beta\gamma$, est en même temps hauteur et bissectrice du triangle ABC, lequel est alors isocèle.*

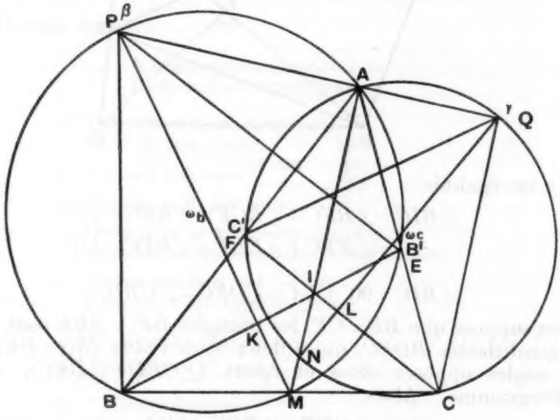


FIG. 2.

3. Considérons encore un triangle quelconque ABC et les bissectrices intérieures BE, CF des angles B, C, qui se coupent en I. Les cercles (w_b), (w_c) circonscrits aux triangles BAE, CFA, rencontrent la bissectrice intérieure AI de l'angle A aux milieux M, N des arcs BEM, CNF, et les points M et N sont tous deux sur le prolongement du segment rectiligne AI, au-delà de I. La bissectrice extérieure de l'angle A des triangles ABC, BAE, CFA, coupe (w_b), (w_c) aux

* V. Thébault, *Education mathématique*, XXXII, p. 57.

milieux P, Q des arcs BPE, CFQ. Ainsi, les segments rectilignes MN, NQ sont les diamètres des cercles (w_b), (w_c) qui coupent les bissectrices BE, CF en K, L, et on a

$$\left.\begin{aligned} MB^2 = MK \,.\, MP = MI \,.\, MA = ME^2, \\ NC^2 = NL \,.\, NQ = NI \,.\, NA = NF^2. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(\text{i})$$

Si $BE = CF$, les cercles (w_b), (w_c) sont égaux ainsi que les triangles isocèles MBE, NFC, et la relation (i) donne

$$MI \,.\, MA = MB^2 = NC^2 = NI \,.\, NA,$$

ce qui prouve que $M \equiv N$.

La bissectrice AI constitue donc l'axe de symétrie de la figure formée par les deux cercles égaux (w_b), (w_c). Les cordes AB, AC de ces cercles étant également inclinées sur l'axe de symétrie, sont égales et le triangle ABC est isocèle. V. THÉBAULT.

SIR THOMAS MUIR.

THE historian of the theory of determinants died at his home near Cape Town on 21st March last, aged 89. Sir Thomas Muir was always a friend of the Association, which he joined on its formation in 1871, and he corresponded regularly with Mr. Greenstreet for many years. At his suggestion and with his encouragement Mr. Greenstreet compiled single-handed the *Catalogue of Current Mathematical Journals* which the Association issued in 1913. Fragments of correspondence which survived among Mr. Greenstreet's papers reveal the immensity of a task which ought to have had the resources of some national institution behind it. Rendered hopelessly out of date by the War within a few months of publication, and superseded presently by the *World List*, this catalogue is now only a domestic monument to the vision and industry of two patient workers.

To the Library Muir gave copies of all his books. In 1926 he sent a set of offprints of his papers, as complete as he could then compile: some 250 papers, spread over more than half a century. He was active till the end of his life, and his later writings have come in unbroken sequence.

It was in 1892 that Muir left Edinburgh, to Scotch the educational system of the Cape, not to kill it, as a friendly wit declared. After forty years abroad he was only a name to most English mathematicians; to the few who have visited Rondebosch perhaps the most vivid memory is of a voice whose lovely tones in everyday speech were a perpetual joy.

E. H. N.

THE TRACING OF CUBIC CURVES.

By E. H. Neville.

1. We all know that if we wish to sketch a cubic curve such as

$$xy(x+2y-1)=2x-3y+4,$$

the first thing to do is to draw the three asymptotes and the satellite. These divide the plane into eleven compartments, and the simple consideration that there can be no points of the curve in a compartment throughout which $xy(x+2y-1)$ and $2x-3y+4$ have different signs enables us at once to rule out six of the compartments, and so to see how the curve goes to infinity. Also the curve does not go through either of the points denoted in Fig. 1 by O, A, B, and does

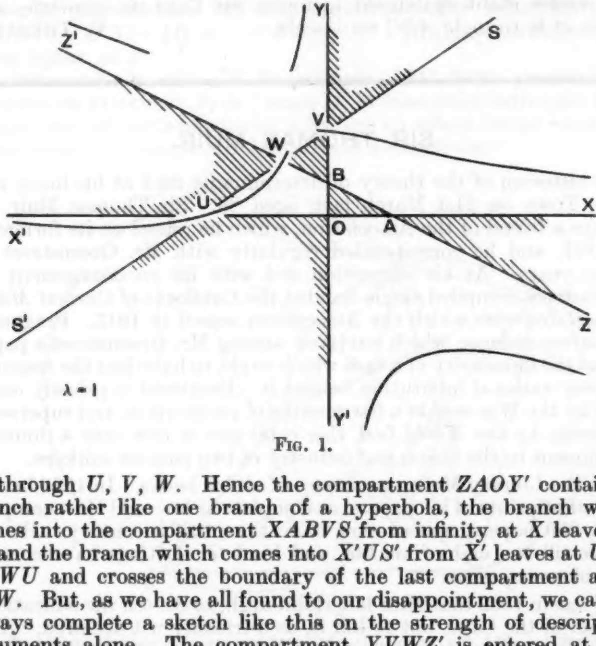


Fig. 1.

go through U, V, W. Hence the compartment $ZAOY'$ contains a branch rather like one branch of a hyperbola, the branch which comes into the compartment $XABVS$ from infinity at X leaves at V, and the branch which comes into $X'US'$ from X' leaves at U for $OBWU$ and crosses the boundary of the last compartment again at W. But, as we have all found to our disappointment, we cannot always complete a sketch like this on the strength of descriptive arguments alone. The compartment $YVWZ'$ is entered at four points: are we to join V to W, leaving a "hyperbolic" branch from Y to Z', or to send the branch XV off to Y and the branch $X'UW$ off to Z', or to cross over from V to Z' and from W to Y?

The cubic

$$xy(x+2y-1)=-(2x-3y+4),$$

which is confined to the complementary set of compartments in the same diagram, presents the same problem. There is a simple branch

from X to Z in the compartment XAZ, but the compartment $X'UWZ'$ is entered not only from infinity at X' and Z' but also at U from Y' and at W from Y by way of V; see Fig. 2. This cubic

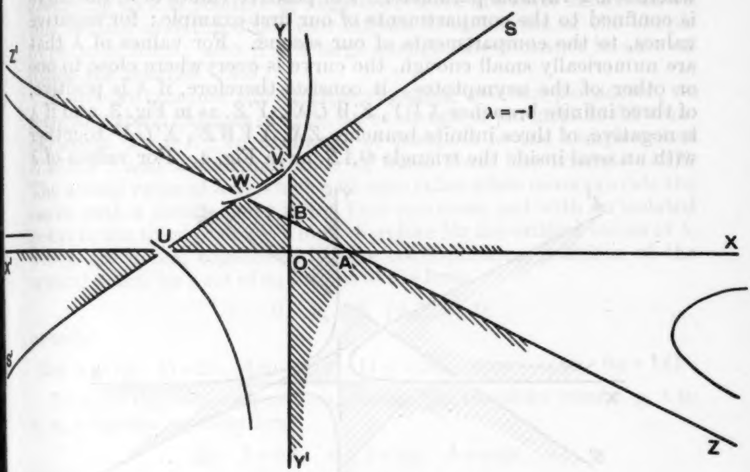


FIG. 2.

presents also a different problem, equally insoluble on general grounds: the compartment OAB is not excluded, but the curve does not cross its boundary, and we have to determine whether or not there is an oval inside this triangle, detached altogether from the infinite branches.

It is easy to accumulate numerical evidence. In these cubics, for an assigned value of one coordinate the other coordinate is given by a quadratic equation; if this equation has roots, definite points of the curve are known, and if not, their absence may effect a discrimination between different types of curve. For example, putting $y=3/2$ in our first cubic we find $3x^2+2x+1=0$; this does not give any points on the curve, but since it proves that the curve cannot pass from either V or W to Y, we know now that the curve consists of a serpentine branch $XVWUX'$ and two hyperbolic branches $Y'Z$ and YZ'. Similarly, putting $y=1/2$ in the second example we have $x^2+4x+5=0$; since then the curve cannot pass from W to either U or X', the infinite branches are $YVWZ'$ and $X'UY'$ in addition to XZ. But no amount of negative evidence will prove that there is not * an oval inside OAB. In view of this failure, and of a suspicion that we might not always be lucky enough to discriminate so easily between the different arrangements of the infinite branches, it is worth while to have a more direct process of investigation.

* With reference to one of his own examples (Pl. 10, Fig. 2), a curve of degree six, Frost can say only "I find no isolated oval, after trying several search-lights".

Consider the curve

$$xy(x+2y-1)=\lambda(2x-3y+4),$$

where λ is a variable parameter. For positive values of λ, the curve is confined to the compartments of our first example; for negative values, to the compartments of our second. For values of λ that are numerically small enough, the curve is everywhere close to one or other of the asymptotes; it consists therefore, if λ is positive, of three infinite branches XVY, $Z'WUX'$, $Y'Z$, as in Fig. 3, and if λ is negative, of three infinite branches ZX, $YVWZ'$, $X'UY'$ together with an oval inside the triangle OAB, as in Fig. 4. For values of λ

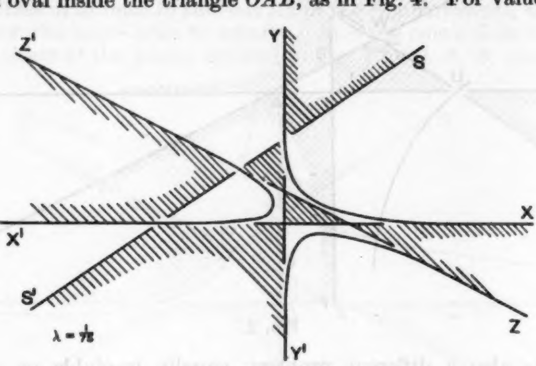


FIG. 3.

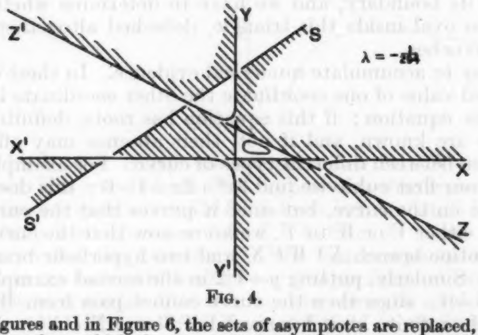


FIG. 4.

In these figures and in Figure 6, the sets of asymptotes are replaced, except near OAB, by curves from which they are indistinguishable, but it must be understood that it is the asymptotes, not the curves, which form the boundaries of excluded regions.

that are numerically large, it is only for values of x and y of which one at least is large compared with λ that the curve is close to an asymptote, while for values of x and y which are small compared with λ the curve is close to the satellite; in this case therefore the

points V, W, U are on a serpentine branch, which is from X to X' or from Y to Y' according as the curve is admitted into the regions $XABVS$ and $X'US'$ or into the regions SVY and $S'UOY'$, that is, according as λ is positive or negative; there are three inflexions on this branch and the other two infinite branches are of the simple hyperbolic form; also if λ is large enough, there cannot be an oval inside the triangle OAB.

If now we consider the curve as changing with the parameter, we recognise three changes of type, an exchange of partners in the compartment $YVWZ'$, an exchange of partners in the compartment $Z'WUX'$, and a disappearance of ovals from the triangle OAB. The actual value of λ at which a change takes place must provide the curve with a double point in the first two cases and with an isolated point in the third case. We look therefore for the critical values of λ, which are given, together with the corresponding positions of the critical point, by a set of equations of the form

$$\phi_x = 0, \quad \phi_y = 0, \quad [\phi_z]_{z=1} = 0,$$

namely,

$$2xy + y(2y-1) = 2\lambda, \quad 4xy + x(x-1) = -3\lambda, \quad -xy = \lambda(4x - 6y + 12).$$

To solve this set of equations, change the variables from x, y, λ to u, v, μ by the substitutions

$$2y - 1 = ux, \quad x - 1 = vy, \quad \lambda = \mu xy,$$

giving

$$2 + u = 2\mu, \quad 4 + v = -3\mu, \quad 2\mu(2x - 3y + 6) + 1 = 0.$$

Since $(2-uv)x = 2+v$, $(2-uv)y = 1+u$, it is evident in advance that μ is to be found from a cubic equation, and therefore that there is only one critical change of each kind, and that no irrelevant values of μ are introduced. For the actual solution, we have

$$2 - uv = 2(3\mu^2 + \mu - 3), \quad (2-uv)(2x-3y) = -(12\mu+1),$$

giving

$$2(12\mu+1)(3\mu^2+\mu-3) - 2\mu(12\mu+1) = 0,$$

with the roots $-1/12$, -1, $+1$. The critical values of λ are $-2/441$, $3/4$, $-5/4$, and the corresponding points are $(2/7, 4/21)$, $(-1/2, 3/2)$, $(-5/2, 1/2)$, of which the first is an isolated point and the others are double points.

The discrimination is now complete, for all values of λ. If λ is positive and greater than $3/4$, the curve has the form which we have already found for the particular value 1, and consists of a serpentine branch $XVWUX'$ and two hyperbolic branches $Y'Z$ and YZ'. If λ is positive and less than $3/4$, the branches are XVY, on which there is one inflexion, $Z'WUX'$, on which there are two inflexions, and $Y'Z$, which is hyperbolic. If λ is negative and numerically less than $2/441$, the curve consists of an oval inside OAB, a hyperbolic branch ZX, a branch $YVWZ'$ on which there are two inflexions, and a branch $X'UY'$ on which there is one inflexion. For values of λ between $-2/441$ and $-5/4$ there is no oval, but the infinite branches

are of the same kind as for smaller negative values of λ; in particular, this is the form of the curve for the value -1. If λ is negative and numerically greater than $5/4$, a serpentine branch $YVWUY'$ separates two hyperbolic branches ZX and $Z'X'$; there is no oval, as is obvious since a line through A which cut such an oval would cut the curve in five points.

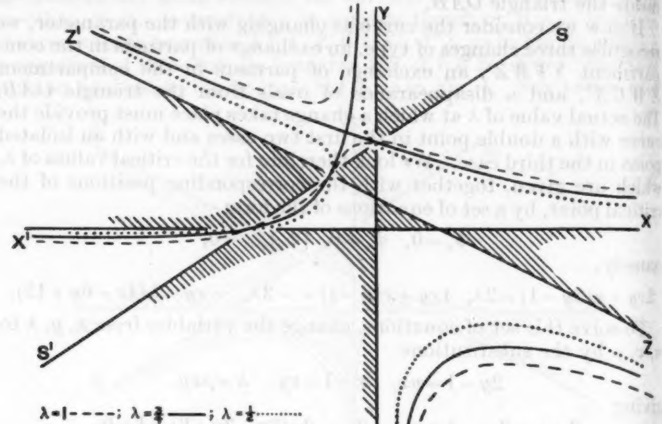


FIG. 5.

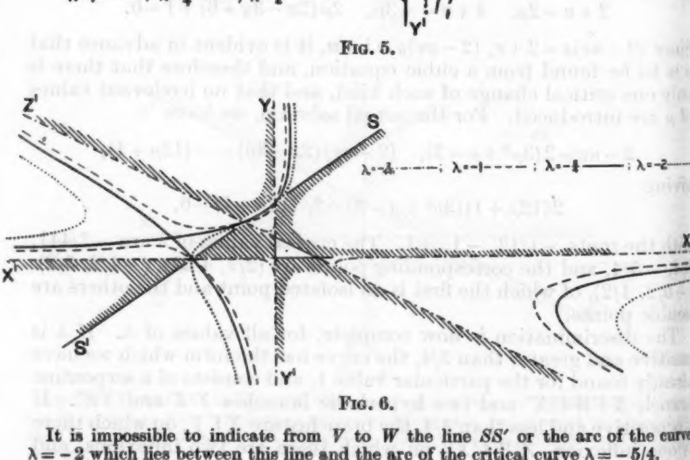


FIG. 6.

It is impossible to indicate from V to W the line SS' or the arc of the curve $\lambda = -2$ which lies between this line and the arc of the critical curve $\lambda = -5/4$.

Even if our object is nothing more than the tracing of one particular curve, the determination of the critical curves may be helpful. The parameter λ may be regarded as a function of position in the plane, determinate except at U, V, W; in other words, curves

associated with different values of λ cannot intersect except at these points, and any curve which is drawn guides to some extent any other curve for which λ has the same sign. The curves with singular points are the easiest to draw, since they are unicursal. Moreover, the determination of the isolated point (2/7, 4/21) is specially useful in the case of a curve with an oval, for this point must be inside the oval, and we can therefore draw lines which we know will cut the oval; in particular, each of the lines through this point parallel to an asymptote, that is to say, each of the three lines $x=2/7$, $y=4/21$, $x+2y=2/3$, cuts the oval in two points given by a quadratic equation. To calculate limits within which k must lie in order that the line $y=k$ should cut an oval belonging to the curve

$$xy(x+2y-1)=\lambda(2x-3y+4)$$

would involve the numerical solution of a biquadratic equation.

It must not be supposed that the process of investigation and solution is peculiar to the example chosen, in any respect except that irrationals have been avoided. The general cubic with three real asymptotes belongs to a family whose equation may be taken as

$$xy(ax+by+c)=\lambda(px+qy+r),$$

the axes being, as a rule, oblique. For the singular points in this family,

$$2axy+y(by+c)=p\lambda,\quad 2bxy+x(ax+c)=q\lambda,\quad cxy=\lambda(2px+2qy+3r),$$

and the substitution

$$by+c=2aux,\quad ax+c=2bvy,\quad \lambda=2ab\mu xy$$

reduces the discrimination to the solution of a cubic equation in μ. If the vertices of the asymptotic triangle are all on the same side of the satellite, as in our numerical example, there are three singularities and the cubic equation has three real roots. If two of the vertices are on one side of the satellite and the third on the other side, the cubic equation has only one real root and there is only one singularity: no oval is possible, and there is only one exchange of partners; in fact in one of the two sets of compartments no ambiguity at all is possible, and the form of the curve can be predicted without calculation, but we have to realise that the simplicity of this case, though not a numerical accident, does not imply that calculation in other cases could be avoided.

2. The curve whose equation is

$$(x^2-2xy+2y^2)(21x-12y-58)=21x-110y+40$$

has only one real asymptote, and the only compartments given immediately by the equation are the four angular regions into which the plane is divided by this asymptote ZWZ' and the satellite SWS'. Since the factor $x^2-2xy+2y^2$ cannot be negative, the curve is confined to the compartments ZWS' and $Z'WS$, and it crosses from one to the other of these at W.

We cannot pretend to sketch the curve from such scanty infor-

mation. It is, however, evident that for any positive value of λ, the curve

$$(x^2-2xy+2y^2)(21x-12y-58)=\lambda(21x-110y+40)$$

lies in the same compartments, and that if λ is very small the curve is close to the line ZWZ', which it crosses at W by making a treble swerve. If λ is very large, the curve is close to the line SWS' except at a great distance from the origin. It is easy to visualise a curve changing from one form to the other by a simple increase in the size of the waves on the two sides of W, and we may be tempted to assume that this is what happens as λ increases. But consider the curves for which λ is negative, which lie in the complementary compartments. If λ is negative and very small, the curve consists of a serpentine branch close to ZWZ' together with an oval surrounding O. As λ increases, the crest of the wave in $Z'WS'$ comes nearer to the oval, until for some critical value of λ there must be a double point; the curve then throws a loop round O. It is only for values numerically greater than this critical value that the curve, if λ is negative, can resume the form which it has for small positive values, namely that of a single branch swerving through W. A misgiving is now inevitable. If the arm of the curve in the region $Z'WS'$ does not pass from WZ' to WS' without passing through a critical form, why should there not be critical forms in the other angular regions also?

The set of equations determining a critical value of λ and the coordinates of the corresponding critical point is

$$2(x-y)(21x-12y-58)+21(x^2-2xy+2y^2)=21\lambda,$$
$$(x-2y)(21x-12y-58)+\ 6(x^2-2xy+2y^2)=55\lambda,$$
$$-29(x^2-2xy+2y^2)=\lambda(21x-110y+60).$$

Introducing variables u, v, μ by the conditions

$$(x-y)(21x-12y-58)=u(x^2-2xy+2y^2),$$
$$(x-2y)(21x-12y-58)=v(x^2-2xy+2y^2),$$
$$\lambda=\mu(x^2-2xy+2y^2),$$

we have

$$2u=21\mu-21,\quad v=55\mu-6,\quad \mu(21x-110y)=-(60\mu+29);$$

if now we put $x-y=ut$, $x-2y=vt$, and substitute

$$x=(2u-v)t=-(34\mu+15)t,\quad y=(u-v)t=-\tfrac{1}{2}(89\mu+9)t,$$

in the two relations

$$(21x-12y-58)t=(x^2-2xy+2y^2),\quad \mu(21x-110y)=-(60\mu+29),$$

we find for μ the cubic equation

$$(60\mu+29)(4181\mu^2+720\mu+783)-116\mu(4181\mu+180)=0,$$

which has the three real roots $1, 87/185, -87/452$. The corresponding critical points are $(1, 1)$, $(65/37, 160/111)$, $(416/339, -200/339)$, and the values of λ are $1, 4205/4107, -26912/38307$.

The negative value of λ necessarily corresponds to the curve which throws a loop round O, and it is easy to see that the point $(292/339, -200/339)$ is in fact in the region $Z'WS'$. For negative

values of λ which are numerically smaller than this, the curve consists of a branch which is nowhere far from ZWZ', together with an oval round O. For negative values of λ which are numerically larger than this critical value, the curve consists entirely of one branch, which passes between O and WS' before turning down towards the asymptote WZ'.

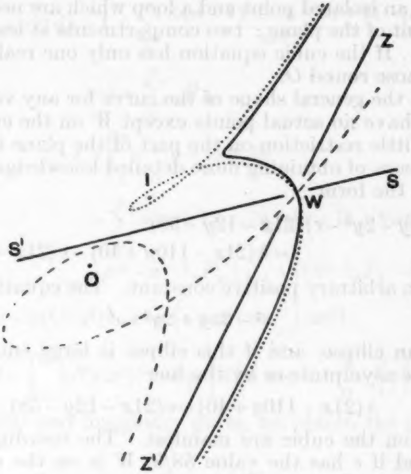


FIG. 7.

The critical curves of a family with one asymptote. The isolated point I belongs to the curve without a double point. The isolated point O and the asymptote ZZ' compose the degenerate member of the family.

For positive values of λ smaller than 1, the curve consists of one branch. When λ is equal to 1, the branch has no obvious peculiarity, but the point (1, 1) is an isolated point whose coordinates satisfy the equation. For values of λ between 1 and 4205/4107, the curve has an oval round (1, 1) in addition to the infinite branch. For the next critical value, the curve has a double point at (65/37, 160/111) and has a loop inside which the ovals for the values of λ greater than 1 are nested. Lastly, for positive values of λ larger than 4205/4107, the curve again consists of only an infinite branch, but this passes from W between WS' and the loop of the critical curve before returning towards the asymptote WZ.

It will be noticed that there are no singular points of the family in either of the regions SWZ, $Z'WS$. The family is not in this respect unduly simple. On the contrary, since the determination of critical values depends on a cubic equation, no family, whatever the numerical values of the coefficients, can have more than three singularities in addition to the isolated point at O. One of these singularities must be the double point on the curve which has a loop

round O, and must therefore be in the same compartment as O. It is obvious that just as the existence of O implies the existence of a double point on some loop round O, so the existence of another isolated point implies the existence of another double point and conversely; that is to say, if the cubic equation has all its roots real, the two which are not associated with the loop round O are associated with an isolated point and a loop which are necessarily in one compartment of the plane: two compartments at least must be free from ovals. If the cubic equation has only one real root, the only ovals are those round O.

Although the general shape of the curve for any value of λ is now known, we have no actual points except W on the curve, and comparatively little restriction on the part of the plane in which it lies. A simple means of obtaining more detailed knowledge is to write the equation in the form

$$(x^2-2xy+2y^2-c)(21x-12y-58)$$
$$=\lambda(21x-110y+40)-c(21x-12y-58),$$

where c is an arbitrary positive constant. The equation

$$x^2-2xy+2y^2=c$$

represents an ellipse, and if this ellipse is large enough to be cut either by the asymptote or by the line

$$\lambda(21x-110y+40)=c(21x-12y-58),$$

restrictions on the cubic are manifest. The coordinates of W are (10/3, 1), and if c has the value 58/9, W is on the ellipse and the ellipse must be effective. Having drawn one ellipse, it is easy to anticipate the result of contracting or expanding the ellipse and rotating the line $\lambda(21x-110y+40)=c(21x-12y-58)$ simultaneously. Instead of actually drawing a number of ellipses we may draw a number of parallel lines and one ellipse, for it is evident that if W_c is the point in OW such that $OW_c/OW=3\sqrt{c}/\sqrt{58}$, a point in which the line $\lambda(21x-110y+40)=c(21x-12y-58)$ cuts the ellipse $x^2-2xy+2y^2=c$ is on the same radius through O as a point in which the parallel line through W_c cuts the similar ellipse through W.

A word may be said in conclusion on the algebraical transformations by which the critical points of a family have been found. Both in the case of three asymptotes and in the case of one, the origin of coordinates has been taken at a special point. But whereas in the latter case no assumption was made as to the axes, in the former the axes were taken to be the two asymptotes whose intersection marks the origin. If the asymptotes are given separately by factors with rational coefficients, this amounts only to a preliminary change of variables from x, y to l_1x+m_1y, l_2x+m_2y, and undoubtedly economises computation in the long run. But the steps used in the case of the unreal asymptotes are of course equally effective if the asymptotes are real, and if the complete factorisation introduces irrationals, it may well be worth while to find a critical equation without a preliminary change in the coordinate system. E. H. N.

MATHEMATICAL NOTES.

1112. ***Two definite integrals.***

The following results may be of some slight interest. Let n be an integer. Then

$$\int_0^\infty e^{-x}(1+x)^n dx = 1 + \sum_1^n \frac{n(n-1)\dots(n-m+1)}{m!} \cdot \Gamma(m+1)$$

$$= 1 + \sum_1^n {}_nP_m.$$

Thus
$$I = \int_0^\infty e^{-x}(a+ix)^n dx$$

$$= \sum_0^n n(n-1)\dots(n-m+1) \cdot i^m a^{n-m}$$

$$= a^n - a^{n-2}\,{}_nP_2 + \dots + i[a^{n-1}\,{}_nP_1 - a^{n-3}\,{}_nP_3 + \dots].$$

But $a+ix = (a^2+x^2)^{\frac{1}{2}} \cdot \exp\left(i \arctan \frac{x}{a}\right)$ and

$$I = \int_0^\infty (a^2+x^2)^{n/2} \exp\left\{i\left(n \arctan \frac{x}{a} - \frac{x}{i}\right)\right\} dx.$$

Equating real and imaginary parts, we obtain the pretty, if useless, integrals

$$\int_0^\infty (a^2+x^2)^{n/2} e^{-x} \cos\left(n \arctan \frac{x}{a}\right) dx = a^n - a^{n-2}\,{}_nP_2 + \dots$$

and

$$\int_0^\infty (a^2+x^2)^{n/2} e^{-x} \sin\left(n \arctan \frac{x}{a}\right) dx = a^{n-1}\,{}_nP_1 - a^{n-3}\,{}_nP_3 + \dots.$$

A. F. Mackenzie.

1113. ***The equations for the foci of a conic.***

If the foci are defined as points from which the tangents are isotropic, the equations satisfied by their coordinates are easily found to be

$$\frac{X^2-Y^2}{a-b} = \frac{XY}{h} = S$$

where

$$X \equiv ax+hy+g, \quad Y \equiv hx+by+f.$$

These equations may be solved as follows. The equation

$$X^2-Y^2 = (a-b)S$$

easily takes the form

$$C(x^2-y^2) - 2Gx + 2Fy + A - B = 0,$$

while $XY = hS$ is equivalent to

$$Cxy - Fx - Gy + H = 0.$$

If $z=x+iy$ these two equations may be combined in the form

$$Cz^2-2(G+iF)z+(A-B)+2iH=0, \quad \text{.................(i)}$$

or
$$\{Cz-(G+iF)\}^2=-\{(AC-G^2)-(BC-F^2)\}+2i(FG-CH)$$
$$=-\Delta\{(a-b)-2ih\};$$

that is,
$$(z-z_1)^2=-\frac{\Delta}{C^2}re^{2i\theta}$$

where

$$z_1=(G+iF)/C, \quad r^2=(a-b)^2+4h^2, \quad \tan 2\theta=-2h/(a-b). \text{ ...(ii)}$$

Hence
$$z=z_1\pm\frac{\sqrt{(-\Delta r)}}{C}e^{i\theta}. \text{ ....(iii)}$$

In numerical cases it is usually best to start from (i). Thus for the conic

$$8x^2+4xy+5y^2-36x-18y+9=0$$

we have

$$A=-36, \quad B=-252, \quad C=36, \quad F=36, \quad G=72, \quad H=144,$$

and (i) becomes

$$z^2-(4+2i)z+6+8i=0,$$

that is,
$$\{z-(2+i)\}^2=-(3+4i)=(1-2i)^2.$$

Hence
$$z=(2+i)\pm(1-2i)$$
$$=3-i \quad \text{or} \quad 1+3i,$$

and the real foci are $(3,-1)$ and $(1,3)$.

R. C. J. HOWLAND.

1114. *Stokes's integral theorem: a direct consequence of integrating the conjugate differential dyadic.*

Any dyadic ϕ can be divided into a self-conjugate and an anti-self-conjugate dyadic as follows:

(1) $$\phi=\tfrac{1}{2}(\phi+\phi_c)+\tfrac{1}{2}(\phi-\phi_c)$$

where ϕ_c is the conjugate of ϕ; the first term on the right-hand side the self-conjugate part, and the second term, the anti-self-conjugate part of ϕ.* By dot multiplication with the radius vector $\bar{r}$ we have

(2) $$\phi\,.\,\bar{r}=\tfrac{1}{2}(\phi+\phi_c)\,.\,\bar{r}+\tfrac{1}{2}(\phi-\phi_c)\,.\,\bar{r}$$

but
$$\tfrac{1}{2}(\phi-\phi_c)\,.\,\bar{r}=-\tfrac{1}{2}\phi_v\times\bar{r}\,\dagger$$

and

(3) $$-\tfrac{1}{2}\phi_v\times\bar{r}=-\tfrac{1}{2}(I\times\phi_v)\,.\,\bar{r}\,\ddagger$$

where ϕ_v is the vector of the dyadic ϕ and I is the idemfactor.

Substituting these results into (2) gives

$$\phi\,.\,\bar{r}=\tfrac{1}{2}(\phi+\phi_c)\,.\,\bar{r}-\tfrac{1}{2}(I\times\phi_v)\,.\,\bar{r},$$

* Gibbs-Wilson, *Vector Analysis*, Yale Press, 1901, p. 296.

† *Op. cit.* p. 297.

‡ Weatherburn, C. E., *Advanced Vector Analysis*, G. Bell & Sons, 1924, p. 91.

from which we obtain

(4) $$\phi = \tfrac{1}{2}(\phi + \phi_c) - \tfrac{1}{2}(I \times \phi_v).$$

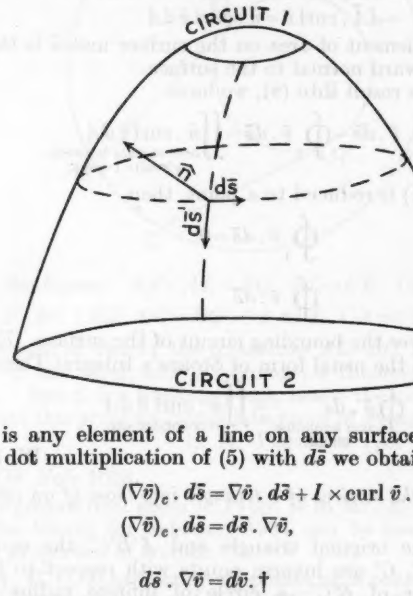
Let $\bar{v}$ be any single valued, finite and continuous vector point function. It follows that $\nabla\bar{v}$ is a dyadic point function and, therefore, can be expressed by (4) as follows :

$$\nabla\bar{v} = \tfrac{1}{2}[\nabla\bar{v} + (\nabla\bar{v})_c] - \tfrac{1}{2}I \times \text{curl } \bar{v}. *$$

Solving for $(\nabla\bar{v})_c$, we have

(5) $$(\nabla\bar{v})_c = \nabla\bar{v} + I \times \text{curl } \bar{v},$$

which is the expression of the conjugate differential dyadic.



If $d\bar{s}$ is any element of a line on any surface, as in the figure, then by dot multiplication of (5) with $d\bar{s}$ we obtain

(6) $$(\nabla\bar{v})_c \,.\, d\bar{s} = \nabla\bar{v} \,.\, d\bar{s} + I \times \text{curl } \bar{v} \,.\, d\bar{s},$$

but $$(\nabla\bar{v})_c \,.\, d\bar{s} = d\bar{s} \,.\, \nabla\bar{v},$$

and

(7) $$d\bar{s} \,.\, \nabla\bar{v} = d\bar{v}. \dagger$$

Substituting these results into (6) gives

$$d\bar{v} = \nabla\bar{v} \,.\, d\bar{s} + I \times \text{curl } \bar{v} \,.\, d\bar{s}.$$

By integrating this equation about any closed circuit on the surface, the left-hand side becomes zero, and the equation reduces to

$$0 = \nabla \oint \bar{v} \,.\, d\bar{s} + \oint I \times \text{curl } \bar{v} \,.\, d\bar{s}.$$

* Weatherburn, *op. cit.* p. 129.

† *Ibid.* p. 120.

Rearranging the above equation and dot multiplying it by another element of a line $d\bar{s}'$ on the surface we get

$$d\bar{s}' . \nabla \oint \bar{v} . d\bar{s} = -d\bar{s}' . \oint \operatorname{curl} \bar{v} \times d\bar{s}.$$

By applying (7) to the left-hand side we get the following:

$$d \oint \bar{v} . d\bar{s} = -d\bar{s}' . \oint \operatorname{curl} \bar{v} \times d\bar{s}.$$

Integrating the above equation from circuit 1 to circuit 2 we obtain

$$(8) \qquad \oint_2 \bar{v} . d\bar{s} - \oint_1 \bar{v} . d\bar{s} = -\iint_{\substack{\text{over surface between}\\ \text{circuits 1 and 2}}} d\bar{s}' . \operatorname{curl} \bar{v} \times d\bar{s}$$

but
$$\begin{aligned} -d\bar{s}' . \operatorname{curl} \bar{v} \times d\bar{s} &= d\bar{s}' \times d\bar{s} . \operatorname{curl} \bar{v} \\ &= d\bar{A} . \operatorname{curl} \bar{v} = \hat{n} . \operatorname{curl} \bar{v}\, dA \end{aligned}$$

where $d\bar{A}$ is any element of area on the surface and $\hat{n}$ is the corresponding unit outward normal to the surface.

Substituting this result into (8), we have

$$(9) \qquad \oint_2 \bar{v} . d\bar{s} - \oint_1 \bar{v} . d\bar{s} = \iint_{\substack{\text{over surface between}\\ \text{circuits 1 and 2.}}} \hat{n} . \operatorname{curl} \bar{v}\, dA$$

Now if circuit (1) is reduced to a point, then

$$\oint_1 \bar{v} . d\bar{s} = 0$$

and
$$\oint_2 \bar{v} . d\bar{s}$$

is an integration over the bounding circuit of the surface. Equation (9) then reduces to the usual form of Stokes's Integral Theorem

$$\oint_{\substack{\text{over bounding}\\ \text{circuit}}} \bar{v} . ds = \iint_{\text{over surface cap.}} \hat{n} . \operatorname{curl} \bar{v}\, dA$$

W. SOLLER.

1115. *To invert the vertices of a triangle into those of an equilateral triangle.*

Let ABC be the original triangle and $A'B'C'$ the equilateral triangle. Then B', C' are inverse points with respect to the perpendicular bisector of $B'C'$—a circle of infinite radius passing through A'. But a circle and two inverse points invert into a circle and two inverse points. Hence the preceding invert into a circle through A with respect to which B and C are inverse points, and on which the centre of inversion lies. This circle is the Apollonius circle through A.

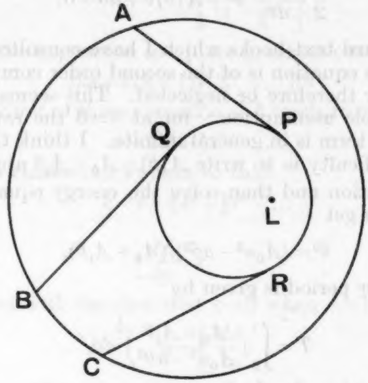
Hence the two positions in the plane of the centre of inversion are the two points of intersection of these Apollonius circles; or, in space, replacing these circles by spheres, the circle at right angles to the plane and having these points as extremities of a diameter.

S. H. MOSS [per N. M. GIBBINS].

1116. *Purser's theorem.*

If AP, BQ, CR are tangents to a circle and the sum of two of the quantities $AP \cdot BC$, $BQ \cdot CA$, $CR \cdot AB$ is equal to the third, then the circle touches the circumcircle of ABC.

Proof. It is known that if L is a limiting point of the coaxal system set up by two circles and PT is a tangent from a point on one circle to the other, then the ratio $PT : PL$ is constant.



Hence in the figure, $AP : AL = BQ : BL = CR : CL$.

Hence $AP \cdot BC : AL \cdot BC = BQ \cdot CA : BL \cdot CA = CR \cdot AB : CL \cdot AB$.

Hence the sum of two of the quantities $AL \cdot BC, BL \cdot CA, CL \cdot AB$ is equal to the third, and by Ptolemy's theorem A, B, C, L are concyclic. But L is a limiting point, hence the circles touch.

Note that this gives an immediate proof of Feuerbach's theorem.

S. M. Plotnick [per N. M. Gibbins.]

1117. *On Note* 1098.

A short geometrical proof of Prop. II in Mr. A. G. Walker's note 1098 in the *Gazette* for February 1934 can be found based on the following properties of a conic :

(i) if A, B, C, D are concyclic points on a conic the bisectors of the angles between AB and CD are parallel to the axes ;

(ii) the bisectors of the angles between the tangents drawn to a conic from any point are the same as those of the angles between the lines joining the point to the foci.

For, let tangents be drawn to the envelope from the centre of the conic S and let these tangents meet S in A, B and C, D respectively. Then AB and CD subtend right angles at O and therefore A, B, C, D lie on the circle passing through O and concentric with S.

Prop. II now follows in virtue of the properties (i) and (ii) above.

S. G. Horsley.

1118. ***Small oscillations of a body with one degree of freedom.***

If the energy equation of a body with one degree of freedom is approximately

$$\tfrac{1}{2}A(\theta)\dot{\theta}^2+\tfrac{1}{2}a\theta^2=\text{constant},$$

near an equilibrium position $\theta=0$, then by differentiation, we get

$$\frac{1}{2}\frac{dA(\theta)}{d\theta}\dot{\theta}^3+A(\theta)\ddot{\theta}+a\dot{\theta}=0.$$

All the standard text-books which I have consulted say that the first term in this equation is of the second order compared with the others, and may therefore be neglected. This seems to me to be a quite unjustifiable assumption; for at $\theta=0$ the ratio of the first term to the last term is in general infinite. I think the best way to get over this difficulty is to write $A(\theta)=A_0+A_1\theta$ approximately in the energy equation and then solve the energy equation. If $\dot{\theta}=\omega$ at $\theta=0$, we then get

$$\dot{\theta}^2=(A_0\omega^2-a\theta^2)/(A_0+A_1\theta),$$

whence a quarter period is given by

$$T=\int_0^{\alpha}\left(\frac{A_0+A_1\theta}{A_0\omega^2-a\theta^2}\right)^{\frac{1}{2}}d\theta,$$

where $$a\alpha^2=A_0\omega^2.$$

Hence we have approximately

$$T=\int_0^{\alpha}\left[\frac{1+\frac{1}{2}(A_1/A_0)\theta}{\omega^2-(a\theta^2/A_0)}\right]^{\frac{1}{2}}d\theta$$

$$=\frac{\pi}{2\omega}+\frac{A_1\omega}{2a}.$$

This shows that if ω is small, which means the same as α small, then the A_1 term can be neglected compared with the A_0 term; or $dA/d\theta$ can safely be neglected throughout the whole oscillation.

H. V. Lowry.

1119. ***The definition of the logarithm.***

The fundamental property of a system of logarithms is that they are auxiliary numbers which proceed in arithmetic progression from 0 when the numbers to which they are attached increase in geometric progression from 1 upwards, so that the scaffolding on which the system is based is

Number -	- 1,	r,	r^2,	r^3,	r^4,	...,
Logarithm -	- 0,	k,	$2k$,	$3k$,	$4k$,	... .

It is necessary to see how to fill in the gaps, which is the same thing as finding what sort of series the logarithms form when the numbers run in arithmetic progression.

Thus let x, x' be two numbers r^n, r^{n+h} of the number series and y, y' be the corresponding logarithms nk, $(n+h)k$,

then $$x'-x=r^n(r^h-1)$$

and $$y'-y=kh.$$

Thus $$\frac{x'-x}{y'-y}=r^n\left[\frac{r^h-1}{kh}\right]$$
$$=\frac{x}{k}\cdot\left[\frac{r^h-1}{h}\right]$$

If $r=1+s$, the expression in square brackets can be expanded as
$$\{hs+h(h-1)s^2/2!+\ldots\}/h\,;$$
the suggested limit of this expression as h tends to zero is very clearly
$$s-\tfrac{1}{2}s^2+\tfrac{1}{3}s^3-\ldots.$$

Hence if we choose our system so that
$$k=s-\tfrac{1}{2}s^2+\tfrac{1}{3}s^3-\ldots,$$
we get
$$\frac{dx}{dy}=x.$$

This, combined with the fact that $y=0$ when $x=1$, gives us
$$y=\int_1\frac{dx}{x},$$
so that the logarithm of a number may be thought of as its entropy.

Now it is easy to calculate logarithms from this, by quite crude methods, to a very fair accuracy. Thus, to get $\log_e 2$, divide the interval from 1 to 2 into 10 equal parts, so that $dx=\cdot 1$. Build up an approximation by dividing each value of dx by the value of x at the beginning of the interval, and we get, as a superior approximation,
$$\tfrac{1}{10}+\tfrac{1}{11}+\tfrac{1}{12}+\ldots+\tfrac{1}{19}=A,\ \text{say}.$$
Similarly, we get an inferior approximation by dividing each interval by the value of x at the end of that interval giving
$$B=\tfrac{1}{11}+\tfrac{1}{12}+\tfrac{1}{13}+\ldots+\tfrac{1}{20}.$$
Between these values lies
$$\tfrac{1}{2}(A+B)=\tfrac{1}{20}+\tfrac{1}{11}+\tfrac{1}{12}+\ldots+\tfrac{1}{19}+\tfrac{1}{40}$$
which is a better value, though still in excess, as, considering the interval as an area, we are taking in place of each strip a trapezium whose upper boundary lies above the curve.

Now, for the sake of using Simpson's rule, take the middle ordinates of each section, giving
$$M=\frac{1}{10\cdot 5}+\frac{1}{11\cdot 5}+\ldots+\frac{1}{19\cdot 5}$$
and we get by Simpson the approximation
$$\tfrac{1}{3}\left[\tfrac{1}{2}(A+B)+2M\right],$$
which gives the logarithm correct to six decimals, with an error of less than 2 in the 7th place.

Comparing the Simpson approximation to $\log(1+a)$, namely

$$\frac{a}{6}\left[1+\frac{8}{2+a}+\frac{1}{1+a}\right]$$

with the series, the error begins with a term in a^5 and is

$$\frac{a^5}{120}-\frac{a^6}{48}+a^7\left(\frac{1}{6}-\frac{1}{7}+\frac{1}{6\,.\,2^4}\right)-a^8\left(\frac{1}{6}-\frac{1}{8}+\frac{1}{6\,.\,2^5}\right)+a^9\left(\frac{1}{6}-\frac{1}{9}+\frac{1}{6\,.\,2^6}\right)\dots$$

and so is easily computed if a is given. Thus, taking $a=\cdot 1$, using the first two terms the error is approximately $(\cdot 1)^6/16$. This gives the error in the first strip of the area. The error in the second strip is less, if we apply Simpson to each strip and *then* add the results.

Thus for the second strip the area is

$$\int_{1\cdot 1}^{1\cdot 2}\frac{dx}{x}=\int_{1}^{12/11}\frac{dx}{x}$$

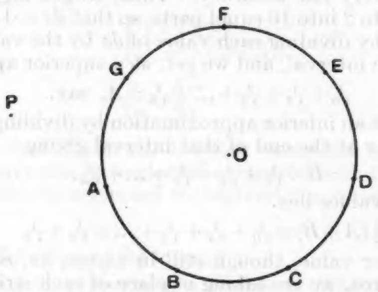
and a is 1/11 and not 1/10, and so on in succession, the error in each strip decreasing till in the final strip we have

$$\int_{1\cdot 9}^{2\cdot 0}\frac{dx}{x}=\int_{1}^{20/19}\frac{dx}{x}$$

and a is only 1/19; its fifth power is about 4 in the 7th decimal, and when divided by 120 will not affect the 8th decimal.

A. LODGE.

1120. ***ABCDEFG is a regular heptagon in a circle of unit radius; to prove that $AC+AD-AB=\sqrt{7}$.***



$$\Sigma PA^2=n\,.\,PO^2+\Sigma OA^2.$$

Let P coincide with A, then

$$AB^2+AC^2+\dots+AG^2=7AO^2+OA^2+OB^2+\dots+OG^2.$$

Hence

$$AB^2+AC^2+AD^2=7.$$

Let $AB=b$, $AC=c$, $AD=d$, then this becomes

$$b^2+c^2+d^2=7. \quad\dots\dots\dots\dots\dots\dots\dots\dots\text{(i)}$$

By Ptolemy's theorem for the quadrilateral $ABCE$,

$$AB\,.\,CE+BC\,.\,AE=AC\,.\,BE,$$

or $$bc+bd=cd. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

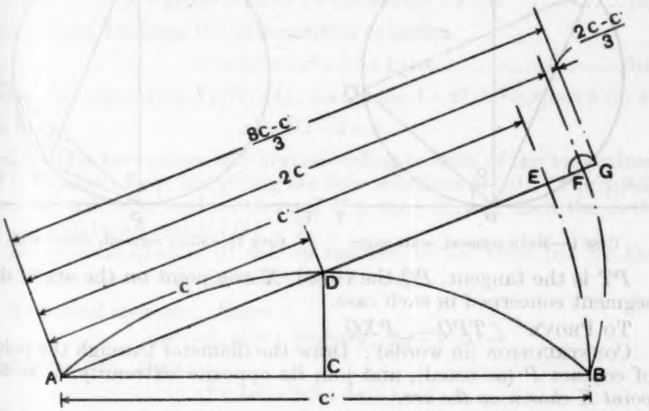
Now
$$\begin{aligned}(c+d-b)^2&=b^2+c^2+d^2+2cd-2b(c+d)\\&=b^2+c^2+d^2 \text{ from (ii)}\\&=7 \quad\text{from (i).}\end{aligned}$$

Thus $$AC+AD-AB=\surd 7.$$

T. S. TUFTON.

1121. ***Construction for the length of a circular arc.***

A simple construction, based on Huygens' formula for the length of a circular arc is obtained as follows. Huygens' formula is $\frac{1}{3}(8c-c')$ where c is the chord of half the arc, and c' is the chord of the whole arc. Putting this in the form $2c+\frac{1}{3}(2c-c')$, the following construction is obtained.



If A and B are the extremities of the arc, join AB and bisect at C. Draw CD perpendicular to AB to cut the arc in D. Join AD and produce. With centre A and radius AB, describe a circle to cut AD produced in E. With centre D and radius DB, describe a circle to cut AD produced in F. Trisect EF and set out FG equal to $\frac{1}{3}EF$. Then AG is approximately equal to the arc.

As the length of the line given by this construction is

$$r\,[\theta-\theta^5/7680\ldots]$$

the error is very small, being less than $\frac{1}{8}\%$ when the angle subtended is 90°. Under the same circumstances the error in Rankin's construction is $0{\cdot}6\%$.

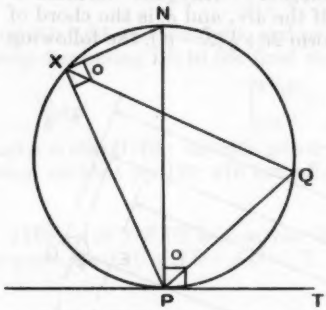
T. W. HALL.

1122. ***Note on the " Alternate Segment " Theorem.***

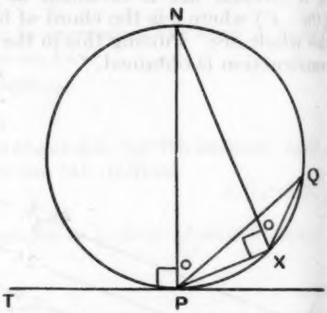
Usual method, other than by limits :

Deal first with the acute angle in the major segment, by drawing the diameter through the point of contact of the tangent and joining its opposite extremity to the other extremity of the chord in the figure, taking the particular angle so formed as " the " angle in the alternate segment ; derive the obtuse angle from this, by using cyclic quadrilaterals.

The following method is, I think, new ; it is certainly shorter by two or three steps, and has the great advantage that proofs for acute and obtuse angles may be obtained independently by parallel proofs, varied only by one alternative step of adding or subtracting a pair of angles for the obtuse or acute angled case respectively. Moreover, *any* position of the angle in each segment may be taken.



CASE I.—Major segment, acute angle.



CASE II.—Minor segment, obtuse angle.

***PT* is the tangent, *PQ* the chord, *X any* point on the arc of the segment concerned in each case.**

TO PROVE : $\angle TPQ = \angle PXQ$.

CONSTRUCTION (in words) : Draw the diameter through the point of contact *P* (as usual), and join its opposite extremity *N to the point X chosen on the arc.*

PROOF : $\angle TPN = \angle PXN$ (tangent perp. to radius ; semicircle).

$\angle TPQ = \angle QXN$ (same segment).

Case I, subtract ; Case II, add. Q.E.D.

To be strictly accurate, *X* is not *any* point in Case I : unless, in that instance, *X and Q are on opposite sides of PN*, the proof becomes cumbersome. But the choice of *X* over the range of the semicircle specified is quite unrestricted.

I met this proof only a few weeks ago, when two boys in the same school examination both independently invented it, having apparently temporarily forgotten the standard method. And, O ye pessimists, the examination was held in a large hall and the two boys were seated at least four rows apart. Honi soit qui mal y pense !

G. H. G.-G.

1123. *Solution of* $x^4+bx^3+cx^2+bx+1=0$.

The following is a simple generalization of the method given by H. Freeman * for the solution of the equation $1+x^4=7(1+x)^4$.

Take the quartic

$$ax^4+bx^3+cx^2+dx+1=0 \quad \text{......................(i)}$$

and write $\quad x=r\cos\phi, \quad 1=r\sin\phi,$

so that $\quad x=r^2\cos\phi\sin\phi,$

and introduce powers of unity so that the r's cancel. Then

$$a\cos^4\phi+b\cos^3\phi\sin\phi+c\cos^2\phi\sin^2\phi+d\cos\phi\sin^3\phi+\sin^4\phi=0,$$

giving

$$a\cos^2\phi+\sin^2\phi+(c-a-1)\cos^2\phi\sin^2\phi+(b+d)\cos\phi\sin\phi$$
$$-\cos\phi\sin\phi(b\sin^2\phi+d\cos^2\phi)=0.$$

This reduces to a quadratic in $\cos\phi\sin\phi$ if $a=1$ and $b=d$, when it becomes

$$(c-2)\cos^2\phi\sin^2\phi+b\cos\phi\sin\phi+1=0, \quad \text{.............(ii)}$$

and (i) then becomes the symmetrical equation

$$x^4+bx^3+cx^2+bx+1=0. \quad \text{....................(iii)}$$

From (ii), $\cos\phi\sin\phi=1/H$, say, and since $1+x^2=r^2=x/\cos\phi\sin\phi$

we have $\quad x^2-Hx+1=0,$

which gives two values of x corresponding to each of the two values of H obtained from (ii), giving the four solutions of (iii). The quadratic (ii) will have real solutions if $b^2 \geqslant 4(c-2)$, and then the roots of (iii) will be all real if $|H| \geqslant 2$.

The general quartic (i) can be reduced to the form (iii) by the substitution $x=a^{-\frac{1}{4}}y$, provided that $b=da^{\frac{1}{2}}$.

Numerical example. Solve

$$4y^4+8\sqrt{2}y^3+10y^2+4\sqrt{2}y+1=0.$$

Put $y=x/\sqrt{2}$; then

$$x^4+4x^3+5x^2+4x+1=0.$$

Equation (ii) is $\quad 3\cos^2\phi\sin^2\phi+4\cos\phi\sin\phi+1=0,$

giving $\quad \cos\phi\sin\phi=-\frac{1}{3} \quad \text{or} \quad -1.$

Then $\quad x^2+3x+1=0 \quad \text{or} \quad x^2+x+1=0,$

giving the four solutions

$$\tfrac{1}{2}(-3\pm\sqrt{5}), \quad \exp\left(\frac{2\pi i}{3}\right), \quad \exp\left(\frac{4\pi i}{3}\right),$$

so that

$$y=\tfrac{1}{4}\sqrt{2}(-3\pm\sqrt{5}), \quad \tfrac{1}{2}\sqrt{2}\exp\left(\frac{2\pi i}{3}\right), \quad \tfrac{1}{2}\sqrt{2}\exp\left(\frac{4\pi i}{3}\right).$$

The original equation $1+x^4=7(1+x)^4$ is the case in which $b=14/3$, $c=7$.

A. F. MACKENZIE.

* *Math. Gazette*, VIII, December, 1916, Note 499, p. 336.

REVIEWS.

La Théorie du Potentiel et ses applications aux problèmes de la physique mathématique. By N. M. Gunther. Pp. 306. 70 fr. 1934. Collection Borel. (Gauthier-Villars)

This book gives a useful account of some important parts of the subject named in the title, from the point of view of the modern analyst. Two extracts will show the lines on which it is written: " Si la densité μ du potentiel de simple couche est une fonction bornée et integrable, le potentiel est continu dans tout l'espace ". " Si le noyau $K_n(1, 0)$ est borné, on peut lui faire correspondre le déterminant de Fredholm $D(\zeta)$, son premier mineur $D(\zeta, 1, 0)$ et la résolvante

$$\frac{D(\zeta, 1, 0)}{D(\zeta)}\text{''}.$$

The book is not meant as an introduction to the theory. From the beginner's standpoint many things are left out. There is no reference to the theory of images and inversion or to the attraction of ellipsoids, and the discussion does not apply to a hemispherical or cubical domain.

The conditions of Liapounoff are assumed throughout and are stated at the beginning of the first chapter, with some of their consequences. In the latter part of the chapter we have the theorems of Green, Stokes and Gauss. The second chapter contains theorems as to the continuity of the potential and the existence of its derivatives for surface and volume distributions. The third and fourth chapters are devoted to the problems of finding functions, harmonic in a given domain, and having on the boundary of that domain

(1) a given normal derivative (Neumann's problem);
(2) given values (Dirichlet's problem).

The integral equations to which these problems lead are treated with much elegance.

The fifth chapter deals with various further developments. Green's functions for the problems of Neumann and Dirichlet are introduced and the solutions by means of the respective Green's functions are given. Then other boundary problems are attacked. The first is that of stationary temperature, the second is that of the equation

$$\nabla^2 U = LU + K,$$

where L, K are known functions and L is positive.

Then comes the wave equation

$$\frac{\partial^2 U}{\partial t^2} = a^2 \nabla^2 U + K$$

and then the equation of the conduction of heat

$$\frac{\partial U}{\partial t} = a^2 \nabla^2 U + K.$$

In each case the usual boundary conditions lead to integral equations, which are treated in various interesting ways.

While the book as a whole is excellent, there are corrections that would make it better. Even on the first page the meaning may be misunderstood. Any sphere satisfies Liapounoff's conditions, and it might be taken that the domain limited by a finite number of intersecting spheres was within the scope intended. This is not so. Again, on p. 116 at line 12 there is an important reference to (21) which should apparently be (71), and on p. 238 at line 8, if the second term on the right is correct, the step is a difficult one.

There is no bibliography and there are no references. A. C. D.

Atomic Theory and the Description of Nature. By N. Bohr. Pp. 119. 6s. 1934. (Cambridge)

Professor Bohr's unique position in quantum theory as himself a discoverer and as an inspirer of discoveries by others is too well known to call for comment. In the present book he figures in the no less important *rôle* of critical commentator on the general progress of the subject. It is a reprint of four essays which appeared originally in *Nature* in 1925 and 1927, in *Die Naturwissenschaften* in 1929, and in *Fysisk Tidsskrift* in 1929, together with an introductory survey which appeared in the Year Book of Copenhagen University for 1929. They therefore supply some of Bohr's contemporary comments on the most critical period in the development of quantum theory from the "old" theory, formulated essentially by himself, to the new Quantum Mechanics in its present form.

Every sentence of the book is couched in a form which immediately commands one's best attention. Professor Bohr speaks with the authority of profound insight and mature consideration of the subject. Nothing he writes is easy to read, but that is because so much thought is condensed into each clause. It amply repays any trouble one may take over reading it for the simple reason that it remains of permanent worth. That, in fact, is the most remarkable feature of the book, that these utterances of Bohr's, made during the days when people jested about "this week's quantum theory", can be reprinted to-day without any suspicion of being out of date. At each stage in the development of the subject he has sensed what is of lasting value and given a masterly appreciation of it. The only fault one can find is that this book leaves one wishing for more. The remedy is, however, promised in the Preface in the form of a further volume of later essays. One can imagine no better way for anyone having some acquaintance with the technique of the subject to acquire its "natural philosophy" than by the study of these works.

It is easier to comment upon the more incidental points of interest than upon the more fundamental. The former include one making special appeal to readers of the *Gazette*, and that is the tribute paid by Bohr to the essential part played by mathematics in the evolution of quantum theory. Another is his careful appreciation throughout of the work of all those who have contributed to the fundamental advance of the subject. In this category of *obiter dicta* one may place, too, some suggestive analogies which he draws between the disturbance of a physical system by the act of observing it, and the alteration of a train of thought when the thinker thinks about (*i.e.* observes) his own thoughts. He touches also upon the question of whether it can ever be possible to give a physical theory of "life", since it may be that the most refined observation of a living organism must necessarily destroy its life.

It is impossible to summarise the fundamental parts of this book without writing something almost as long as the book itself! In fact, on first reading the book for the purposes of this notice, one underlined certain important passages deserving special comment. Then on commencing a second reading one decided that further passages were equally important, until it became evident that to mark fundamentally important statements would amount to underlining almost every word in the book!

The ideas most insisted upon may, however, be briefly mentioned under two aspects:

(i) The quantum theory makes a definite and irrevocable renunciation of the classical formulation of the concepts of *space-time* and *causality* applied to atomic phenomena. This is a consequence of the indivisibility of the quantum of action, which results in a finite disturbance of any system by the act of observing it. This disturbance cannot be allowed for in a causal scheme, for it involves such an interaction between the system observed and the observing mechanism as excludes the possibility of an unambiguous distinction between them. The classical formulation is replaced therefore by what Bohr calls a

"complementarity" or "reciprocity" between the causal and space-time modes of description, which means that the use of one precludes the simultaneous use of the other.

(ii) The classical concepts themselves can nevertheless not be discarded, for all our experience must ultimately be interpreted in terms of them. It therefore emerges that the predictions of quantum theory must take the form of *statistical relationships* between the results of measurements of physical phenomena. In view, however, of some misleading popular discussions it is important to realise, as Bohr points out, that "there is no question of a failure of the general fundamental principles of science within the domain where we could justly expect them to apply". Nor is there any lack of precision in the mathematical methods, only the interpretation of their results is brought more into consonance with the form in which the physical data are presented. Rather is the position that summed up by Bohr when he says, "Perhaps the most distinguishing characteristic of the present position of physics is that almost all the ideas which have ever proved to be fruitful in the investigation of nature have found their right place in a common harmony without thereby having diminished their fruitfulness". W. H. McCrea.

The Progress of Science. By J. G. Crowther. Pp. x, 304. 12s. 6d. 1934. (Kegan Paul)

This book is not in any sense a systematic survey of modern developments in science. It is rather a series of essays on selected topics suggested to the author, apparently, by his recent travels and reading. Physics he treats variously from the points of view of work done in the Cavendish Laboratory, in Copenhagen, and in Soviet Russia, together with chapters on "The Stars and the Universe", "Cosmic Rays", and "Diplogen". Biological subjects are dealt with in chapters on "The Chemistry of Evolution", "Human Heredity", and "Pernicious Anaemia".

Whatever else he has done, Mr. Crowther has produced a book with the initial merit of being highly interesting. For example, he tells in a lively manner how the patient following up of slight clues led to the discovery of "cosmic" rays, and how the great body of subsequent refined observation has led to results of fundamental importance including the discovery of the positive electron. Or again, he shows how the systematic study of the chemistry of eggs of different animals has yielded new ideas of first-rate importance in the theory of evolution. He deals also with a number of interesting side issues, as when he mentions that the Russian mathematical physicist Landau is engaged in the reconstruction of the course in mathematical physics at Karkov to produce "a method of presenting the appropriate matter from a modern point of view, and eliminating methods and subjects which remain in the course only through historical inertia". One would like to learn the results of this effort.

In a book which ranges from the explanation of the permanent waving of ladies' hair to that of the expansion of the universe one would be surprised to discover no slips. As a matter of fact there are a number, of varying degrees of seriousness. They range from the careless use of terms, like *energy* instead of *momentum* (p. 41), *force* instead of *energy* (p. 47), to confusions of principle as when on p. 50 the author states that radioactivity follows from the Principle of Uncertainty, which he explains as meaning that the position of a particle cannot be measured beyond a known limit of accuracy. Actually radioactivity appears as a consequence of the wave character of the propagation of matter without any explicit appeal to the uncertainty principle. Further, this principle does not mean that the position of a particle cannot be known to any desired degree of accuracy, but sets an upper bound to the accuracy with which its position and momentum may be known *simultaneously*. Again, the account of the expansion of the universe contains many flaws. The author says, for

example, that " if there were no radiation, no expansion would occur " (p. 103), but this is wrong on all existing theories. For another thing, he treats Milne's theory (p. 108) as though it offered a physical explanation different from that of general relativity, instead of being largely a different mathematical procedure. The result of these mistakes is that a reader familiar beforehand with some of the subjects will be chary of accepting the author's assertions on subjects with which he is not familiar, while the reader without previous knowledge will imbibe a mixture of true and false ideas, without being able to know which is which.

The author's intention is " to help the general reader to increase his knowledge of science and to make him more impatient of leaders who cannot take a scientific view of human problems ". In helping to stimulate an interest in the aims of modern science he should have attained a good measure of success. But in producing the ability to think scientifically, or in communicating accurate scientific knowledge, his success is more doubtful. This is partly on account of the type of error already mentioned. It is mainly, however, because these ends cannot be attained by means of popular accounts such as this book provides, but only through the discipline of a full scientific training.

W. H. McCrea.

The Sub-Atoms, an interpretation of spectra in conformity with the principles of mechanics. By W. M. Venable. Pp. viii, 248. 9s. 1933. (Williams and Wilkins, Baltimore; Baillière, Tindall and Cox)

Sub-atoms, according to the author, are units, each consisting of a negative electron and a positive particle of certain size, shape, and mass, out of which the atoms of the elements are built up. Thus, while the hydrogen atom is taken as identical with sub-atom 1, the helium atom is assumed to be a composite body formed of sub-atom 2 together with two sub-atoms 1, the lithium atom to be formed of sub-atom 3 together with two sub-atoms 2, and so on. The radiation of such atoms is supposed to be produced by the electron "bouncing" upon the surface of its accompanying positive particle. This gives the continuous spectrum. The line spectrum is supposed to be due to a complicated process of " co-operation " between atoms and arrangements of molecules.

This is not mathematical physics; it is " may-and-might-ical " physics. It is a description of the consequences which, in the opinion of the author, "may", " might", or " are to be expected to " follow from a host of hypotheses. It is prompted by a desire to explain atomic phenomena on Newtonian mechanics alone, without recourse to any quantum postulate whatever. It was, however, the failure of such attempts that led to the discovery of the quantum theory. It is still, of course, legitimate to examine the reasons for such failure. But actually there is no particular reason for expecting success, and it is certainly not profitable to substitute for the quantum postulate innumerable *ad hoc* hypotheses on atomic constitution in order to make Newtonian theory work in this domain. Surely any other system of mechanics could, by such means, be made to serve equally well. In the author's case he finds it necessary to introduce one new sub-atom, with suitable assumed properties, for each different element dealt with. It is difficult to see what advantage this offers beyond saying that a given element has certain characteristics merely because it is that element!

Actually the assumed properties do not appear to explain even those phenomena for whose explanation they were invented. For example, using faulty electrodynamics, the author deduces that the bouncing electron will emit a continuous range of frequencies bounded on the high frequency side. He concludes, " Our model is thus even capable of explaining, to some extent, continuous spectra, which for the most part are . . . neglected by the popular (*sic*) theories . . . ". The actual fact is that the " popular " theories give a very

precise explanation of the continuous spectra, which are, however, unfortunately for the new theory, bounded in general on the side of *low* frequency and not of high frequency.

It would be a serious omission to conclude a notice of this sort without a reference to the recent commentary by R. H. Fowler (*Nature*, June 9, 1934, p. 852), on all efforts such as the work under review. He gives an impressive survey of the way in which quantum mechanics has provided a rational basis for the whole of physics, with the possible exception of some nuclear phenomena and of gravitation. This achievement would have to be surpassed by any rival theory intended to supplant quantum theory. No successful rival has yet appeared.

W. H. McCrea.

Higher Mathematics for Engineers and Physicists. By I. S. and E. S. Sokolnikoff. Pp. vi, 482. 24s. 1934. (McGraw-Hill)

This book comes from the University of Wisconsin, and, according to the preface, is the substance of an "orientation" course of lectures offered annually to engineering students, and which has latterly found favour among an increasing number of students of physics and chemistry. Those of us who are responsible for the mathematical education of such students will recognise the problem which the authors are attempting to solve, namely the difficulty which these students find in increasing their mathematical attainments by reading books "written primarily for mathematicians". The aim has been to produce a volume which will "bridge the gap which separates many engineers from mathematics . . . and serve as a stepping-stone to advanced mathematical treatises".

Let it be said at the outset that the exposition is very clear, and adequate, that a large number of carefully chosen exercises (with answers) are included, and that the reviewer's failure to find any typographical errors is a further proof of the care which has so evidently been expended upon this volume. From the purely mathematical standpoint, too, fault is hard to find. Difficulties, such as those connected with convergence, and with double limit problems, are not slurred over, and where they are not completely resolved (and the more intricate proofs are frequently—and, in our opinion, wisely—withheld) the theorems, and their limitations, are clearly and explicitly stated. Nor can it be said that any engineering or science student would not be well advised to master the subject-matter of the various chapters, or that, given the will to do so, he could fail to follow and appreciate the sweet reasonableness of the argument.

From the standpoint of the English engineering student, however, as far as he is known to the present writer, it cannot be said that the book solves the problem stated above. Much of the subject-matter would normally be met with in his undergraduate course in mathematics (*i.e.*, solution of equations, determinants, partial differentiation, Fourier series, ordinary differential equations, and much of the chapter on empirical formulas and curve-fitting). He might also have a nodding acquaintance with improper integrals, multiple integrals, some of the properties of infinite series, and some of the vector analysis. This leaves the chapters on elliptic integrals, line integrals, partial differential equations, probability, and a final chapter which is a first introduction to conformal representation, as what may be to him new matter. In his post-graduate mathematical requirements, this book would not be a "bridge", but merely the first stepping-stone. Nor can it be said that the *real* problem of the engineer who finds he *needs* mathematics, namely the translation of engineering data into mathematical language and equations, is even touched upon. The writers are too essentially mathematicians—and, in fact, their sureness of touch occasionally falters over points in applied mathematics.

Finally, however good an American book—and this has distinct excellences—one cannot recommend its purchase, so long as American publishers refuse

to face economic facts, and fix their English prices on a long out-of-date dollar-pound exchange rate. W. G. B.

Malerische Perspektive. I. By K. Bartel. Translated from the Polish (1928) by W. Haack. Pp. viii, 339. Geb. RM. 16. 1934. (Teubner)

The first of two volumes on the art of graphical or applied perspective deals mainly with linear perspective. But perspective, as known to geometers, is a comparatively recent development in the history of art. Then are the paintings which survive at Pompeii, is early Christian art to be considered as non-perspective, or can there be two or more different species of perspective? Within what province does the artist's perspective lie—that of the geometer, that of the physiologist or that of the psychologist? These are among the many questions which the author proposes to answer by studying the development of perspective in graphical art from the earliest ages down to the present day.

Circumstances have unfortunately delayed the production of the second volume, and judgment on the full value of the work must be reserved. The first volume is largely a gathering together of evidence; the reasoning out of the case will follow in the second.

But judged on its own merits, the present volume deserves the highest praise. The very fact that it is designed to fit into a larger scheme lends it a fascination in striking contrast to the aridity of the usual self-contained treatises on perspective. The reader will find a geometrical diagram, complete with lines, planes and vanishing point, facing the reproduction of a Dürer engraving. He will see the elliptic projection of a circle illustrated by a photograph of the circus at Warsaw with its ring and tiers of seats. He will realise that everything that is artistically best, and most evidently a manifestation of genius, whether best by reason of its pleasing the eye, or best by reason of its suitability to its purpose, in short the essence of pictorial or constructive art, depends in some essential point upon a strict and perhaps toilsome obedience to geometrical laws. Conversely, he will deplore the modern demand for results without effort, for "stunts", for short cuts, and its cult of brilliance without basis. That particularly stupid protest: "But what can Science, what in particular can Mathematics, have to do with Art", is countered by the words of Leonardo da Vinci: "Those who devote themselves to Practice without Theory are as mariners who go to sea in ships without rudder or compass".

The book is uniform with the author's *Kotierte Projektionen* reviewed recently (XVIII, p. 132) and is of the same high standard of production. The first six chapters are concerned with free or direct perspective, the perspective of the artist. Chapters V and VI, which deal respectively with reflected images and shadows, are particularly interesting. The final chapter VII treats of *Gebundene Perspektive*, the methods by which a perspective drawing is obtained, not directly as in free perspective, but indirectly through the medium of some descriptive representation such as plan and elevation. It concludes with a description of the perspectograph of C. de la Fresnaye (1909).

The mere fact that applied geometry has been so shamefully neglected does not in the least diminish its importance. Sooner or later a revival of interest will come, and if Vol. II of Bartel's *Malerische Perspektive* fulfils the promise of Vol. I, the work will be quoted as a classic. Therefore the second volume must not be without a comprehensive index to the whole. E. L. I.

Differential Equations. By N. B. Conkwright. Pp. xii, 234. 7s. 6d. 1934. (Macmillan)

This little book, based upon a course given by the author at the University of Iowa, contains more than is regarded as adequate for the highest Honours

in provincial Universities. It is a compact and cheap presentation of the usual matter in the usual manner, with references to sources and indications for further reading. It is a little curious to find oneself referred for information on certain topics to text-books little larger or no more advanced than this, which is as honest and as unbusinesslike as the shop which will sell you shirts, socks and ties, but refers you for collars to the rival establishment next door.

Taken as a whole it is a good elementary account. The first 132 pages are devoted to ordinary equations, then 15 to numerical approximations, and 42 to partial equations. The remainder is made up of appendices (including an unnecessary 16 pages on Standard Integrals), answers to between five and six hundred examples, and an index.

There is perhaps a slight tendency to formulate rules for wresting a solution out of an equation, but on the whole the methods are sound. In particular, full use is made of symbolic operators. The author has a great fondness for footnotes, but slips on p. 30. Considering its comparative cheapness, the book is reasonably well produced. The printers, however, were not very successful with fractional indices.

If there is any meaning in the threat that " no part of this book may be reproduced in any form without permission in writing from the publisher, except by a reviewer who wishes to quote brief passages in connection with a review written for inclusion in a magazine or newspaper ", it is that no other author will now be able to write an elementary account of differential equations.

E. L. I.

Differential Equations. By L. R. Ford. Pp. x, 263. 15s. 1933. (McGraw-Hill)

So many new text-books on differential equations have come on the market in recent years that each newcomer has to justify its appearance by displaying novelties or showing improvements which distinguish it from its predecessors.

L. R. Ford's treatment aims at sharpening the geometrical or intuitive presentation by gradually mingling with it the rigorous, a mode of approach which, it is said, has been rather neglected in American text-books. The early parts of the text aim at providing an effective introduction to the subject; the later parts at furnishing material for a second course in the calculus suited to students who have gained some facility in its methods and applications. If the opportunity occurs, the whole may be merged into a single year's course. The result is a sandwich. Chapters I to V form one fairly homogeneous slice, VI and VII (numerical methods) provide the something different in the middle, and VIII to XI repeat, in form if not in matter, the texture of the first slice.

Like a sandwich, too, the book is admirably fitted to the purpose for which it was designed. The first three chapters, which cover 81 pages, provide a clear and well-illustrated account of what differential equations are, how they arose, and how the commoner types are integrated. The introduction naturally takes the geometrical point of view, for at the early stages that point of view is the most stimulating and the most fruitful. In the author's words, " this manner of presentation, properly used, is capable of creating much enthusiasm for the subject ". It would be quite unreasonable to expect any very novel matter in these chapters, but, on the other hand, the treatment might be attractive or not. Here the style is pleasing—simple and direct. The only point which appears to be somewhat of a blemish on an otherwise precise and lucid account occurs in the treatment of the reduced linear equation with constant coefficients. In saying " The solution is arrived at by an ingenious device. We try a solution of the form $y=e^{mx}$ ", the author commits the crime of urging the trial of a special dodge because he and his fellow experts know that it will work. The fewer of the " ingenious devices " the better.

Chapter IV, *The Method of Successive Approximations* (pp. 82-92), opens with an instructive example illustrating the step-by-step development of a solution with assigned initial conditions. This serves to bridge the gap between the intuitive methods which have preceded and the rigorous treatment which is to follow. The usual questions of existence, uniqueness, and dependence on a parameter and on initial values are presented in a form suited to the understanding of the students who, fresh from a first course in the calculus, may have begun to imagine that all mathematics is manipulation. It may console Professor Ford to learn that he is not the first to have mis-stated the Lipschitz condition; actually he uses it in the correct form. In Chapter V (pp. 93-112) extension is made to bring in systems and equations of higher order and to introduce the integrable total differential equation in three variables.

Chapter VI, *Interpolation and Numerical Integration* (pp. 113-146) and VII, *The Numerical Solution of Differential Equations* (pp. 147-162), break the sequence of the theoretical work of the book. It is hard to understand why they should have been intruded at this precise point; a better course would have been to place them at the end of the book, and to leave the reader to consult them at any stage that he or his instructor deemed to be the most convenient. Apart from this, there is a great deal in Chapter VI (undoubtedly of importance in itself, and certainly well presented) that is out of place in a discussion of differential equations, and is not necessary for an understanding of Chapter VII. The method of that chapter, based upon Simpson's rule, is attributed to W. E. Milne; the details were worked out by the author himself. It was tested and, apart from a tedious initial step, found to be satisfactory.

Now follow Chapter VIII, *Linear Equations* (pp. 163-182) and IX, *Certain Classical Equations* (pp. 183-204). This is a meagre allowance of space for so vast and important a section of ordinary differential equations. Nevertheless, Chapter VIII does furnish a sound introduction to the general theory of the linear equation and IX is devoted to solutions in series, with special reference to the hypergeometric equations and the equations of Legendre and Bessel. The discussion of the functions thus defined (in particular that of the roots of the Legendre polynomials and Bessel functions) is a vast improvement on the tedious and fruitless collection of formulae usually found in elementary text-books. On the other hand, it practically ignores the second solution of the Legendre equation and does not even hint that the Bessel equation has a solution other than $J_n(x)$. Solutions by definite integrals are not mentioned.

Chapter X, *Partial Differential Equations of the First Order* (pp. 205-232) is undoubtedly one of the most instructive introductory accounts available. Geometrically based, it seems to bring the hidden mechanism of a partial differential equation to light in a remarkable manner. The non-linear case is discussed first; the linear considered as a degenerate case. The concluding chapter (pp. 233-257) considers some of the simplest partial differential equations of the second order, with one or two applications, which is all that space allows.

Apart from a certain lack of homogeneity in design, this is one of the best productions on the market. It should prove to be a valuable class text-book, the pace is even and the style is fluent. There is an adequate number of suitable examples.

E. L. I.

Adolf Hurwitz. Mathematische Werke. I. Funktionentheorie. Pp. xxiv, 734. 1932. II. Zahlentheorie, Algebra und Geometrie. Pp. xiv, 755. 1933. Schwfr. 80; geb. schwfr. 88. (Birkhäuser, Basel)

The average mathematician can rarely afford to acquire the collected works of his gods; but while he may console himself with the thought that the memorials of an Euler or a Cauchy are almost too voluminous for his private

shelves, it is hard indeed to resist the desire to possess some of the smaller but equally attractive crystallisations of mathematical ability.

Adolf Hurwitz (1859-1919) was a competent mathematical craftsman with a gift for lucid exposition; most mathematicians know his *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen* completed by Courant and published in 1922, and will agree that as a short account of the Weierstrassian conception of the theory of functions and of the main outlines of elliptic functions, Hurwitz' work in this volume is hard to beat. Thence we can realise why the reputation of his teaching at Zürich stood so high, if we remember that at the same time his research work covered almost the whole field of late nineteenth century analysis. Shortly after his death, proposals for an edition of his collected works were made by the department of mathematics in the "Technischen Hochschule" of Zürich, where Hurwitz had spent more than half his working life, but the adverse economic conditions of the post-war epoch delayed the scheme. Recently financial help was forthcoming and publication achieved; thanks for these two noble volumes are due to the various benefactors, to the printers and publishers for their really admirable work, and to the editors, chief among whom is, of course, Professor Pólya.

As a schoolboy Hurwitz learned much from Schubert, and his first paper, published in 1876, was the result of a collaboration with Schubert. For a while he studied at Berlin under Weierstrass and Kronecker, but the dominating influence throughout his career was that of Klein, from whom he acquired a knowledge of Riemann's work on function-theory and an interest in modular functions which was to prove of great value when he turned his attention to the theory of numbers. Other influences were those of Hilbert and Minkowski, with whom he enjoyed an intimate friendship during the eight years he spent at Königsberg; and it is clear from his work that he must have studied the writings of Dedekind with particular attention.

Probably what is most important in Hurwitz' researches is contained in his numerous papers on various aspects of Riemann's theory, and his equally numerous papers on the class number of quadratic forms and kindred topics. But to discuss these in any detail would require more space than is here available. It will perhaps be of more interest to readers of the *Gazette* to add some comments on various groups of papers, interesting in themselves and capable of being read profitably and enjoyably without any very special equipment. All Hurwitz' papers, in fact, bear evidence of having been written up with great care, and his *Inauguraldissertation*, "Grundlagen einer independenten Theorie der elliptischen Modulfunktionen und Theorie der Multiplikator-Gleichungen erster Stufe", can still be warmly recommended as a clear introduction to its subject.

(i) Some papers on the geometrical applications of Fourier series form an interesting excursion into the byways of analysis, though one of the theorems required as a starting point is of great importance to the modern theory. In the first of these papers, Hurwitz observes that if we take a simple closed curve of length 2π and express the coordinates x, y as functions of the arc s, periodic in 2π, and expand them as Fourier series in s, then it rapidly follows that the area of the curve is greater than or equal to π, and further that if the area is equal to π, the series for x and y must reduce to

$$x=a_0+a_1 \cos s+a_1' \sin s, \quad y=b_0-a_1' \cos s+a_1 \sin s,$$

the parametric equations of a circle; hence we have the isoperimetric property of the circle. In making further applications of a like nature, Hurwitz rediscovered the now famous Parseval theorem concerning the Fourier constants of a function and gave a proof of the theorem on the assumption that the function is integrable (in the sense of Riemann); as soon as it was pointed out to him that proofs had been given by Liapounoff to the Mathematical Society of Kharkow and by de la Vallée Poussin in the *Annales de la soc. sci. de Bruxelles* he hastened to give a full acknowledgment of their priority. In Band

57 of the *Annalen* there is another paper on Fourier theory in which Hurwitz makes use of the new ideas introduced by Fejér and Cesàro.

(ii) In 1919 Hurwitz published his *Vorlesungen über die Zahlentheorie der Quaternionen*, the germ of which is to be found in a paper first published in the *Göttinger Nachrichten*. Lipschitz had endeavoured to construct an arithmetic of quaternions, taking, very naturally, the integer quaternions to be those with integer components. This led to difficulties not easily overcome, and Hurwitz was the first to develop a perfect arithmetic of quaternions by taking the integer quaternions to be those whose components were integers or half-integers. The theory is valuable not only for its own elegance, but also for its applications to the theory of numbers. For instance, in the paper referred to, it enables Hurwitz to solve the problem, dating back to Euler:

"Determine all integral linear substitutions

$$y_a = a_{0a}x_0 + a_{1a}x_1 + a_{2a}x_2 + a_{3a}x_3 \quad (a = 0, 1, 2, 3)$$

such that
$$y_0^2 + y_1^2 + y_2^2 + y_3^2 = M(x_0^2 + x_1^2 + x_2^2 + x_3^2),$$

so that the form $x_0^2 + x_1^2 + x_2^2 + x_3^2$ transforms into a multiple of itself".

(iii) In some of his papers, Hurwitz gave new proofs of known results, sometimes results in fairly elementary domains. In the paper "Über die Einführung der elementaren transzendenten Funktionen in der algebraischen Analysis" he shows how the functions of elementary analysis may be defined by an iterative production of monotonic sequences, similar to the Newton iteration for the square root, or Gauss's definition of the arithmetico-geometric mean. He illustrates the method by definitions of the logarithm and the inverse tangent; it is interesting to observe that Professor Landau has adopted this method of defining the logarithm in his recent book on the calculus (see *Gazette*, July 1934, p. 216). Hurwitz also obtains in this paper a fairly general theorem giving conditions under which iterative processes of the type

$$x_n = \phi(x_{n-1}) \quad (n = 1, 2, 3, \ldots)$$

will supply a limit which is a regular function of x_0.

(iv) Another branch of pure mathematics to which Hurwitz turned frequently and successfully was that dealing with the zeros of algebraic and transcendental functions. On this topic he wrote several papers; the first and perhaps the most famous of these is the paper of 1889, "Über die Nullstellen der Bessel'schen Funktion", in which he relies on an important and interesting theorem which can be stated roughly thus: if $\{f_n(z)\}$ is a sequence of functions tending uniformly to a limit function $f(z)$ then the zeros of $f(z)$ are the limiting points of the set of all the zeros of each $f_n(z)$. He applies this principle to the Bessel functions and in a later paper to equations of the type

$$g(z) \sin z + h(z) \cos z = 0,$$

where g and h are rational functions, and to functions given by the series

$$\sum_n \frac{z^n}{\Gamma(n+1)\Gamma(n+a)\Gamma(n+\beta) \ldots \Gamma(n+\kappa)}.$$

The theorem itself has been completed by Jentzch, who made use of the ideas connected therewith in his researches on over-convergence of Taylor series.

There is much more in these volumes to tempt us to further description, but perhaps enough has been said to show how full of interest and value they are to both the teacher and the research worker. It may be added that the first volume includes a portrait of Hurwitz, and short notices by Hilbert and Meissner.

T. A. A. B.

The New Background of Science. By Sir James Jeans. Second edition. Pp. viii, 312. 7s. 6d. 1934. (Cambridge)

The first edition of Sir James Jeans' description of the present situation in

theoretical physics was reviewed in the *Gazette* for October, 1933 (XVII, p. 274). That a second edition was called for only a year after the publication of the first is a remarkable tribute to the author's power of lucid exposition, and he is to be warmly congratulated on the obvious popularity of this work with the general reader. T. A. A. B.

Reversions and Life Interests. By H. J. Tappenden. Pp. xii, 57. 7s. 6d. 1934. (Cambridge)

This work, published for the Institute of Actuaries Students Society, deals with the question of Reversions and Life Interests in its business aspects and does not concern itself with the methods of calculation of the values of the various contingencies which frequently arise. It is, therefore, not of great interest to the readers of this journal. It gives the minimum information about the legal, economic and other questions with regard to official administration which it is necessary for all students to know who have any dealings with the subject. For this purpose it is admirably designed, and will be found useful.

It is admirably printed as it is issued by the Cambridge University Press.
W. S.

Revision Mathematics. By L. Crosland. Pp. viii, 254. With answers, 3s. 6d. 1934. (Macmillan)

This is a collection of questions which have been set in the various School Certificate examinations, the source of each being indicated. Under the main heads, Arithmetic, Algebra, Geometry, the questions are arranged in sets according to the subject matter, each set being preceded by a few worked-out examples. There are revision papers in arithmetic, algebra and geometry, a set of questions in numerical trigonometry, some typical examination papers, and 18 pages of answers which appear to be accurate. A very useful book for the last year of the School Certificate course. W. J. D.

Rapid Arithmetic Calculations. III. Pp. 47, 16. Without answers, 4d. and 6d.; with answers, 6d. and 8d. 1934. (Oxford)

This part of a larger work deals with household and business transactions, and consists mainly of 87 sets of exercises on the application of arithmetic to everyday affairs. There are a few explanatory pages, some, in particular, dealing with mental calculation of cost. One or two exercises need reconsideration, and the answers are not always correct.

Exercise 22/7 reads, " Arrange in ascending order of magnitude $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$ ". Answer, " $\frac{5}{6}$, $\frac{3}{4}$, $\frac{2}{3}$ ". Exercise 23/3 reads, " In ·5056 the figure 5 appears twice. How much greater in value is the first 5 than the second 5 ? " Answer, " 100 times greater ". How many children would say " 99 times greater " ?

The book is neatly printed and well arranged and successfully meets the practical requirements of girls. W. J. D.

Algebra. By H. J. Mann and J. S. Norman. New edition. Pp. viii, 244, 35. 4s. 6d.; without answers, 4s. 1934. (Deane, Year Book Press)

The authors of this text-book, which was first published 12 years ago, have now issued a new edition enlarging its scope up to the School Certificate examination standard. The text is expressed in simple language, generally clear and sound, so far as it goes, and is well illustrated by abundant graduated exercises. The new matter deals with graphs, harder equations and problems, but does not touch logarithms or simple series, and is not free from errors.

On page 220, $2x$ and $-2x$ should be $2x^2$ and $-2x^2$. On page 226 the expression ax^2+bx+c is transformed into

$$a\left(x+\frac{b}{a}\right)^2-\frac{b^2-4ac}{4a},$$

the error appearing twice. W. J. D.

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